1. **Length Contraction** (50 points)

In this problem, we will explore the nature of the function:

\[ y = \sqrt{1 - \frac{v^2}{c^2}} \].

This quantity, sometimes called the “Lorentz Factor” is the factor in Special Relativity by which an astronaut moving by at speed \( v \) ages. That is, I age 1 year while I observe that the astronaut ages \( y \) years.

We start by exploring the behavior of this function for small velocities of \( v \), and those close to the speed of light. To do this, we’ll need to develop a few mathematical tools. If we define \( x = \frac{v}{c} \), the Lorentz Factor can be written as \( y = \sqrt{1 - x^2} \).

a. (10 points) For very small velocities, \( v \ll c \) and we expect \( y \) to be very close to (but slightly less than) 1. Thus we write \( y = 1 - \epsilon \), where \( \epsilon \ll 1 \). Our exercise will be to determine \( \epsilon \). Solve the equation above for \( \epsilon \) in terms of \( x \): start by squaring both sides of the equation, and then recognize that if \( \epsilon \) is small, \( \epsilon^2 \) is tiny, and additive terms involving \( \epsilon^2 \) can be neglected. With the value of \( \epsilon \) in terms of \( x \) in hand, now express the Lorentz Factor \( y \) in terms of \( v \) and \( c \), for \( v \ll c \).

**Answer:** When \( v \ll c \), then \( v/c \ll 1 \), and \( y \) will be very close to unity. We can do an approximation as follows. We know that: \( (1 - \epsilon)^2 = 1 - 2\epsilon + \epsilon^2 \). Suppose that \( \epsilon \ll 1 \). In this case, \( \epsilon^2 \) is an even smaller quantity, and we will make the approximation that it is negligible. Thus \( (1 - \epsilon)^2 = 1 - 2\epsilon \), to a very good approximation. We can turn this around, and write:

\[ \sqrt{1 - 2\epsilon} = 1 - \epsilon. \]

Now,

\[ y = \sqrt{1 - \left(\frac{v}{c}\right)^2} = \sqrt{1 - 2 \times \frac{1}{2} \left(\frac{v}{c}\right)^2}, \]

just the form we have above, for \( \epsilon = 1/2(v/c)^2 \). Thus if \( v \ll c \), then \( \epsilon \ll 1 \), and

\[ y = \sqrt{1 - v^2/c^2} = 1 - \frac{1}{2} \frac{v^2}{c^2}, \]
to a very good approximation.

Incidentally, this is an example of a general rule, which can be demonstrated with something called a Taylor series (which one learns about in calculus), namely:

\[(1 + \epsilon)^n \approx 1 + n\epsilon,\]

whenever |\(\epsilon\)| \(\ll 1\).

Given the hints given in the problem, most students will simply get this right. Two points off for never expressing the final result in terms of \(v\) and \(c\) (i.e., just keeping the result in terms of \(x\)). Five points off for each algebraic error. Full credit for using the tools of calculus, and phrasing the answer in terms of a Taylor expansion, if they explain this clearly.

b. (10 points) Now let’s take the opposite limit, namely speeds very close to the speed of light. This time, we’ll write \(x = v/c = 1 - \alpha\), where now \(\alpha \ll 1\). Plug into the equation for the Lorentz Factor; similar to part a, if \(\alpha\) is small, then \(\alpha^2\) is really tiny and can be neglected. Thus write an expression for the Lorentz Factor in this case.

**Answer:** Now let’s take the opposite limit, where \(v\) is very close to \(c\). In particular, let’s write \(v/c = 1 - \alpha\), where \(\alpha \ll 1\). Note that \(v^2/c^2 = (1 - \alpha)^2 = 1 - 2\alpha\), to a good approximation. Thus:

\[y = \sqrt{1 - v^2/c^2} = \sqrt{1 - (1 - 2\alpha)} = \sqrt{2\alpha} = \sqrt{2}(1 - v/c).\]

Same grading policy as in (a). Four points off for keeping terms of order \((v/c)^2\).

c. (5 points) Now we’re ready to plug in some numbers. Draw a graph of the Lorentz Factor as a function of velocity, where the x-axis ranges from 0 to the speed of light \(c\), and the y-axis ranges from 0 to 1. Plug in many values of \(v\) and calculate the value of the Lorentz Factor, and plot them up. Describe the shape of this function in words. Next, plot on the same graph the two approximations you calculated, for small velocities and large velocities, in parts (a) and (b). Over what range of velocities does each do a decent job of approximating the Lorentz value? That is, over what range does each approximation give answers within 10% of the correct value?

**Answer:** The graph is shown on the next page, showing both the exact formula for the Lorentz contraction, and the two approximations we’ve just worked out. The approximations work impressively well. The overall shape of the curve is that of a piece of a circle, extending from unity at small speeds, to zero at the speed of light. It is within 10% of the approximation worked out in part (a) for speeds less than 0.76c, and with the approximation worked out in part (b) for speeds above 0.65c.

3 points for the figure; take 1 point off if the three curves are not clearly labelled, 1 point off if only one approximation is shown, and 2 points off if neither approximation is shown. One point for any reasonable (1-2 sentence) description of the curves, and two points for any reasonable statement about the range of velocities in which the approximations seem to work. Indeed, even though we ask for the point at which the
approximations are wrong by 10%, any reasonable qualitative statement about the range of validity of the curves gets full credit. Full credit for curves drawn using graphing software, or done by hand. Give no more than 2 points for hand-drawn curves which have little relationship to the true shape of the curves (i.e., where they didn’t calculate any values directly).

d. (10 points) What is the value of $y$ for $v = 0.001c$, $v = 0.6c$, $v = 0.8c$, $v = 0.99995c$, $v = 0.9999995c$? Use the approximations you’ve just developed, as appropriate. If you find yourself rounding off and saying that $y = 1$ or $y = 0$, you’ve rounded too much!

**Answer:** Here we have to calculate the Lorentz factor for various values of $v$. For $v = 0.001c$, we’ll use the approximation of part (a), and for the two values of $v$ very close to the speed of light, we’ll use the approximation of part (b) (that’s what they are for!). For $v = 0.6c$ and $0.8c$, we’ll do the full calculation, and also show the results for the two approximations; we’ll see that both do a pretty good job!
\[ v = 0.001c, \quad y = 1 - 1/2(v/c)^2 = 1 - 1/2 \times 10^{-6} = 1 - 0.0000005 = 0.9999995. \]

\[ v = 0.6c, \quad y = \sqrt{1 - 0.6^2} = \sqrt{1 - 0.36} = \sqrt{0.64} = 0.8. \] For the small-\(v\) approximation, we get \(y = 1 - 1/2(0.6^2) = 1 - 1/2 \times 0.36 = 0.82\), pretty close! For the large-\(v\) approximation, we get \(y = \sqrt{2(1 - 0.6)} = \sqrt{0.8} \approx 0.9\); not as close to the right answer.

\[ v = 0.8c, \quad y = \sqrt{1 - 0.8^2} = \sqrt{1 - 0.64} = \sqrt{0.36} = 0.6. \] For the small-\(v\) approximation, we get \(y = 1 - 1/2(0.8^2) = 1 - 1/2 \times 0.64 = 0.68\), a bit high. For the large-\(v\) approximation, we get \(y = \sqrt{2(1 - 0.8)} = \sqrt{0.4} \approx 0.63\); pretty good!

\[ v = 0.99995c, \quad y = \sqrt{2(1 - 0.99995)} = \sqrt{2 \times 5 \times 10^{-5}} = \sqrt{10^{-4}} = 10^{-2}. \]

\[ v = 0.9999995c, \quad y = \sqrt{2(1 - 0.9999995)} = \sqrt{2 \times 5 \times 10^{-7}} = \sqrt{10^{-6}} = 10^{-3}. \]

2 points for each calculation. No point for a given part if the student simply rounds to \(y = 1\) or \(y = 0\). In the first and last one, their calculator will be challenged if they try to calculate it “exactly”; it is actually more accurate to do it using the approximation. Take off one point in the first, fourth and fifth points if they do not use the relevant approximation, and attempt to do it exactly. Full credit for 0.6c and 0.8c for either approximation used.

e. (15 points) A muon is a particle very much like an electron, only with more mass. Unlike electrons, muons are unstable; they have a half-life of only \(2.2 \times 10^{-6}\) seconds, after which they decay into an electron, a muon neutrino, and an anti-electron neutrino. Very fast-moving muons are produced in the upper atmosphere of the Earth (i.e., 100 km above the Earth’s surface) in collisions with high-energy cosmic rays (produced ultimately from distant supernova explosions), and come whizzing down to be detected on the surface. A typical atmospheric muon is moving at 0.9999995 the speed of light; i.e., \((1 - 5 \times 10^{-7})c\). Taking into account time contraction, how far could such a muon travel before decaying (i.e., before one half-life is over)? Now do the same calculation, not taking into account time contraction. Discuss: does the fact that we observe muons that were produced 100 km away give any support to Einstein’s prediction of time contraction?

**Answer:** In part (d), we calculated the Lorentz contraction factor for this value of the speed; it is \(10^{-3}\). That is, time for the muon progresses 1000 times slower than it does for us. Thus the muon now has a lifetime 1000 times longer, i.e., \(2.2 \times 10^{-3}\) seconds. How far does it go at almost the speed of light? Well, light travels at 1 foot per nanosecond, and its lifetime is 2.2 million nanoseconds, so it goes 2.2 million feet, or about 700 kilometers.

If it weren’t for the effects of time contraction, it would have its usual lifetime of 2.2 microseconds, and would therefore be able to travel \(1/1000\) as far, i.e., about 700 meters.

Thus the observation of muons which have travelled 100 kilometers is a direct demonstration of time dilation at work; it simply would be impossible to observe cosmic ray muons at the Earth’s surface if Einstein was not correct...
8 points for the calculation of the distance the muon could travel before decaying. Give only 2 points if they multiply, rather than divide, by the factor $10^{-3}$. 4 points for the calculation ignoring time contraction. Full credit for any correct units. Full credit for results consistent with part (d), even if the former is incorrect. 3 points for any reasonable discussion (even just a sentence or two) explaining what this says about the veracity of Lorentz contraction. Only give full credit if the statement is consistent with the numbers. That is, if the numbers are incorrect, and state that the muons cannot travel 100 km before decaying, then the statement should definitely say that there is something wrong!

2. What’s at the center of our Galaxy? (10 points)

You observe a star orbiting the center of our Galaxy at a speed of 1,000 km/s in a circular orbit with a radius of 20 light-days. We infer stellar velocity from the Doppler shift of the absorption lines. This gives the velocity along the line of sight, or “radial velocity”. For some stars we can also detect the proper motion on the sky, if the star is moving fast – this is actually seen near the Galactic center. You can learn more about this in Chapter 19.4 of the book. Based on your measurement, what would be your estimate for the mass of the object that the star is orbiting? What kind of an object do you think this is? Give arguments to support your conclusion.

Solution:
The mass enclosed by a circular orbit can be determined knowing the radius of the orbit and the orbital velocity:

$$M = \frac{rv^2}{G}.$$  

We derived this expression in class when we talked about the galactic rotation curve. This is a simple consequence of the Kepler’s third law, or of Newton’s law of gravitation. To use this expression, we need to convert the radius and the speed of the star into meters and meters per second. The speed we find to be $1.0 \times 10^6$ m/s. The radius is a bit more work, since it is given as 20 light-days. There should be 365 light-days in a light-year, so we can convert and find that the orbital radius is $5.5 \times 10^{-2}$ light-years $= 5.5 \times 10^{-2} \times 3 \times 10^8$ m/s $\times 3 \times 10^7$ s $= 5.2 \times 10^{14}$ m. The mass enclosed by the orbit is, therefore, $5.2 \times 10^{14}$ m $(1.0 \times 10^6$ m/s)$^2/(2/3 \times 10^{-10}$ m$^3$ kg$^{-1}$ s$^{-2}) = 7.8 \times 10^{36}$ kg. This is about 4 million solar masses!

What kind of an object can this be? Let’s proceed with the process of elimination. Can this be a star? No, because we know that the maximum mass of a star is 150 solar masses. Can this be many stars, perhaps a cluster of stars? Globular clusters can have up to a million stars in them. This is a bit on the low side, given that a typical star is 0.5 solar masses. But, even if it were a globular cluster, it would be way too compact – we want it to fit inside the orbit of $5.2 \times 10^{14}$ m, which is roughly 0.01 of a parsec. A typical globular cluster would be at least several parsecs in radius (typically, 10 pc). In fact, we know of only one type of object that can accommodate such an enormous mass confined to such a small volume – a black hole! The black hole in the center of our Galaxy is actually quite tiny compared to other galaxies – black holes up to a few billion solar masses are known to exist in the centers of many distant galaxies.

7 points for the calculation, 3 points for the explanation.
3. Expanding Universe, Expanding Oceans (10 points)
The expansion of the universe discovered by Hubble really only takes place on physical scales larger than galaxies, but for the purposes of this exercise, imagine that the proportionality between distance and recession velocity holds on scales comparable to the size of the Earth. Calculate by what distance the Atlantic Ocean (width 7000 km) would grow in a year under the expansion of the universe. Compare with the actual growth, due to plate tectonics (a completely different physical mechanism), which adds up to several inches a year. Which is the faster mechanism on these scales: plate tectonics, or the expansion of the Universe?

Answer: Here we apply the expansion of the universe to the Atlantic Ocean. As the problem states, this is a bit of a mis-use of the Hubble Law, but it will give us a feel for how fast the expansion is. The speed at which a point 7000 km away would recede, according to the Hubble Law, is:

\[ v = H d. \]

To go on, we need to remember that the Hubble Constant needs to be put into useful units. One way to proceed is to remember, from class, that the reciprocal of the Hubble Constant is the age of the Universe, 14 billion years \( \approx 5 \times 10^{17} \text{ sec} \). Thus \( H \) is the reciprocal of this number, or \( 2 \times 10^{-18} \text{ /sec} \). Plugging this in, we find:

\[ v = H d = 2 \times 10^{-18} \text{ sec}^{-1} \times 7000 \text{ km } \times 10^6 \text{ mm/km} = 1.4 \times 10^{-8} \text{ mm/sec}. \]

Over the period of a year, the distance traveled is this times the number of seconds in a year, namely \( 3 \times 10^7 \text{ sec} \), giving a distance of \( 4 \times 10^{-1} \text{ mm} \), or 400 microns. Plate tectonics gives a much larger expansion, of several inches per year. The real point is the different timescales: the universe took over 10 billion years to get as large as it is, while the Atlantic Ocean only took a few tens of millions of years to get this big due to plate tectonics.

Full credit for results in any reasonable units. Full credit for using any reasonable value of the Hubble Constant, even if it isn’t the value calculated previously. Four points off for not comparing with the expansion distance due to plate tectonics. Seven points off for confusion about which distance to use for the Atlantic Ocean.

4. The Shining Sun (30 points)
a. Four hydrogen atoms (total mass = \( 6.693 \times 10^{-27} \) kilograms) fuse to produce one helium atom (mass = \( 6.645 \times 10^{-27} \) kilograms). If one gram of hydrogen is burned in the sun into helium, how much energy will be produced? (5 points)

Answer: In order to calculate the energy produced when one gram of hydrogen is burned in the Sun into helium, we first calculate the energy released in the creation of one helium atom:

\[ \Delta E = m_{4H}c^2 - m_{He}c^2 = (6.693 \times 10^{-27} \text{ kg } - 6.645 \times 10^{-27} \text{ kg})(3.0 \times 10^8 \text{ m/sec})^2 = 4.3 \times 10^{-12} \text{ Joules}. \]

Then the total amount of energy produced if one gram of hydrogen is burned into helium is the amount of energy produced per atom, times the total number of Helium atoms (or equivalently, the number of hydrogen atoms divided by 4) in one gram. The
latter quantity is simply the reciprocal of the mass of the Helium atom (expressed in grams, not kilograms), so we get:

\[
E_{\text{1 gram } H} = N_{He} \Delta E = \frac{1 \text{ gram}}{m_{He}} \Delta E = (6.6 \times 10^{-24})^{-1}(4.3 \times 10^{-12} \text{ Joules}) = 6.4 \times 10^{11} \text{ Joules.}
\]  

(1)

2 points off for calculating quantities for a kilogram of hydrogen, not one gram. One can get confused by the factor of 4 hydrogen atoms per helium atom; two points off for this confusion. Up to four significant figures is OK; the masses of the atoms were given to four significant figures, although strictly speaking, the difference of masses is only good to 2 significant figures. Give only 2 points for calculating the energy of one gram of hydrogen as $1 \text{ gram } \times c^2$. The problem does not specify the units for the answer; anything correct gets full credit.

b. The sun has a luminosity of $4 \times 10^{26}$ Joules per second. It has a mass of $2 \times 10^{30}$ kilograms. If three quarters of the sun’s mass is hydrogen, using the results of question a) above, make a rough estimate of the maximum time the sun could shine at its present luminosity by burning hydrogen into helium. How does this estimate compare with its actual main sequence lifetime? Explain any discrepancies you might find. (15 points)

**Answer:** The maximum energy the Sun could produce over its lifetime by burning hydrogen is its mass of Hydrogen in grams times the energy released in burning one gram of hydrogen:

\[
E_{\odot, H} = (3/4)M_\odot E_{\text{1 gram } H} = (3/4)(2 \times 10^{33} \text{ gm})(6.4 \times 10^{11} \text{ Joules/gm}) \simeq 10^{45} \text{ Joules}.
\]  

(2)

(Note that we converted the mass of the Sun to grams.)

We can therefore estimate the maximum time of Sun could shine at its present luminosity by burning hydrogen into helium to be

\[
t = \frac{E_{\odot, H}}{L_\odot} = \frac{10^{45} \text{ Joules}}{4 \times 10^{26} \text{ Joules/sec}} = 2.5 \times 10^{18} \text{ sec} \simeq 7 \times 10^{10} \text{ years}.
\]  

(3)

The main sequence lifetime of a solar type star is only $10^{10}$ years. But only the Hydrogen in the core of the Sun is hot enough to burn, so the Sun ends its main sequence lifetime when only 10% of its hydrogen has burned to helium.

11 points for doing the calculation correctly; 4 points for giving the correct explanation that the Hydrogen burns only in the core of the star. Two points for recognizing the discrepancy, without giving the correct explanation.

c. When we talk about hydrogen being burned in the Sun, it is tempting to think of the Sun as getting less massive with time. If the Sun is shining at $4 \times 10^{26}$ Joules per second, and all the energy comes from nuclear burning, calculate what percentage of the Sun’s mass will actually be converted into energy after 10 billion years. (5 points)
Solution

The amount of mass converted into energy every second is

\[ m_{\text{converted in 1 sec}} = \frac{E_{\text{released in 1 sec}}}{c^2} = \frac{L_\odot \times 1\text{sec}}{c^2}, \]

where \( L_\odot \) is the Sun’s luminosity.

The amount of mass converted over 10 billion years is then:

\[ m_{\text{converted in } 10^{10} \text{ yrs}} = \frac{L_\odot \times 10^{10} \text{ yrs}}{c^2} = \frac{4 \times 10^{26} \text{ J/s} \times 10^{10} \text{ yrs} \times 3 \times 10^7 \text{ s/yr}}{(3 \times 10^8 \text{ m/s})^2} = 1.3 \times 10^{27} \text{ kg} \approx 10^{-3} M_\odot. \]

So, the Sun will be lighter by less than 0.1% of its mass – hardly a big effect!

*Full credit by doing this in other ways, e.g., using the results from a) and b).*

d. The Sun is also losing mass through a solar wind – a spherical outflow of charged particles that escape from the outermost layers of the Sun. You can think of these particles as boiling off the Sun (there is a very tenuous but hot layer outside the Sun’s atmosphere, called the “corona” from which these particles escape). The rate of particle loss in the solar wind is \( 10^{36} \) protons per second. What is the rate of mass loss in solar masses per year? Compute how much mass the Sun is going to lose to the Solar wind over its 10 billion year life time (in solar masses). How does this compare to the mass lost during the fusion? (5 points)

Solution

The mass lost per second is \( 10^{36} \times m_p = 1.67 \times 10^9 \text{ kg/s} \), where \( m_p = 1.6 \times 10^{-27} \text{ kg} \) is the mass of a proton. That’s 1.6 million tons per second! Per year, this is \( 1.67 \times 10^9 \text{ kg/s} \times 3 \times 10^7 \text{ sec/yr} = 5 \times 10^{16} \text{ kg/yr} \approx 2 \times 10^{-14} M_\odot/\text{yr} \). Over the lifetime of the Sun it will lose \( 2 \times 10^{-4} M_\odot \), or 0.02% of its mass – less than what is lost due to fusion, but not by much!