1. Transits of extrasolar planets (25 points)

A new NASA mission called “Kepler” was launched on March 6th to observe the transits of extrasolar planets. During a transit, a planet goes in front of the star as seen from our vantage point on Earth. This blocks a little bit of light from the star, so an accurate measurement of the change in brightness of the star will show a temporary dimming. The main goal of the Kepler mission is to detect Earth-size planets in the habitable zone around stars like our Sun. In this problem, we will predict what the Kepler mission will observe. Consider a planet with the radius of the Earth, orbiting at 1 AU from a star that has parameters like our Sun. The orbital plane of this planet lies along our line of sight.

a) How long does a transit last (in hours)? (10 points) Hint: the Earth is very far from this planetary system.

Solution:

The duration of the transit is determined by how long it takes the planet to cross the face of the star. Since we are observing this from a vantage point very far away, the duration is just the diameter of the star divided by the orbital speed of the planet (if we were observing this transit closer to the star we would have to worry about the finite angular size of the star, and when exactly the planet starts crossing the limb of the star). We have found the orbital speed in a Keplerian orbit several times before, so let’s quote the relevant formula from homework 2:

\[ v = \sqrt{\frac{GM_\oplus}{r}}. \]
Substituting numbers for the solar mass and 1AU for the distance, we get the orbital speed of 30 km/s (as can be checked with the calculation we did in homework 1 since the orbital parameters here are identical to the Earth). The duration of the transit is then \(2R_\odot/v = 2 \times 7 \times 10^5 \text{ km}/(30 \text{ km/s}) \approx 5 \times 10^4 \text{ s} \approx 13 \text{ hours}.

Full credit for simply quoting the orbital speed from the earlier homework.

b) By how many percent does the brightness of the host star diminish in the middle of the transit? (10 points)

Solution:
In the middle of the transit the planet blocks the area on the surface of the star equal to the cross sectional area of the planet, \(A_{\text{planet}} = \pi R_{\text{Earth}}^2\) (note, that this is not the total surface area of the planet!). The unperturbed brightness of the star is proportional to the area of the face of the star \(B_{*0} \propto A_{*} = \pi R_{*}^2\) (note that this is also cross-sectional area). In the middle of the transit, the area of the face of the star is reduced to \(B_{*1} \propto A_{*} - A_{\text{planet}}\). The fractional change in brightness \(B_{*}\) of the star as seen by us will then be

\[
\frac{\Delta B_{*}}{B_{*}} = \frac{B_{*0} - B_{*1}}{B_{*0}} = \frac{A_{*} - (A_{*} - A_{\text{planet}})}{A_{*}} = \frac{A_{\text{planet}}}{A_{*}} = \left(\frac{R_{\text{planet}}}{R_{*}}\right)^2.
\]

Substituting \(R_{\text{Earth}}\) and \(R_\odot\), we get the change of \((10^{-2})^2 = 10^{-4}\), or 0.01%.

3 points off for using the total surface areas \(4\pi R^2\). 5 points for recognizing how to do the problem but completely failing in the algebra. 2 points off for answers of 99.99% – not the change in brightness, but the final brightness.

c) Consider now the planet the size of Jupiter, orbiting at 0.5 AU from the same star. By how many percent does the brightness of the host star diminish in the middle of the transit by this planet? (5 points)

Solution

We are observing this transit from very far away, so the distance between the planet and the host star will not affect the answer. The difference is just the cross sectional area of the planet. Jupiter’s radius is 11 times the Earth radius, so the answer in b) will increase by \(11^2 \approx 100\). So, the answer is 1% change in brightness. This change can be detected even with the Earth-based telescopes, and this method has been used to find many “hot Jupiters” – gas planets orbiting close to other stars.

2. White Dwarfs and Neutron Stars (30 points)

White dwarfs and neutrons stars are known as “compact objects” – stars where a large amount of mass (near a solar mass) is concentrated in a small volume. Such stars possess exotic properties that allow us to probe the behavior of matter at extreme densities. Consider a white dwarf (WD) with the mass of 1 M_\odot and radius 1 R_{\text{Earth}}, and a neutron star (NS) with a mass of 1.4M_\odot and radius 10 km.

a) Find how much mass is contained in one teaspoon (1cm^3) of material from WD and NS. Express the answer in metric tons.

*Hint: the material can be taken to be at the mean density of these stars (7 points)*
Solution
To calculate the mass of the teaspoon worth of NS material, we start with computing the mean density of the star. We know that $1.4 \, M_\odot$ is confined within a sphere of radius 10 km:

$$\rho_{\text{NS}} = \frac{M_{\text{NS}}}{\text{Volume of NS}} = \frac{1.4M_\odot}{4\pi r^3/3} = \frac{1.4 \times 2 \times 10^{30} \, \text{kg}}{4 \times 3 \times 10^4 \, \text{m}^3/3} = 7 \times 10^{17} \, \text{kg/m}^3.$$

The mass in the teaspoon is then $\rho_{\text{NS}} \times V_{\text{teaspoon}} = 7 \times 10^{17} \, \text{kg/m}^3 \times 10^{-6} \, \text{m}^3 = 7 \times 10^{11} \, \text{kg} \approx 1$ billion tons. For the WD, we use the same formula, but plug in $1M_\odot$ for the mass and 6400 km for the radius. We find:

$$\rho_{\text{WD}} = \frac{M_{\text{WD}}}{\text{Volume of WD}} = \frac{1M_\odot}{4\pi r^3/3} = \frac{1.4 \times 2 \times 10^{30} \, \text{kg}}{4 \times 3 \times 6.4 \times 10^6 \, \text{m}^3/3} = 2 \times 10^9 \, \text{kg/m}^3.$$

The mass in the teaspoon is then $\rho_{\text{WD}} \times V_{\text{teaspoon}} = 2 \times 10^9 \, \text{kg/m}^3 \times 10^{-6} \, \text{m}^3 = 2 \times 10^3 \, \text{kg} \approx 2$ tons.

b) Calculate the mass density of a neutron and compare it to the mean density of NS. A neutron can be considered as a sphere of radius 1 femto-meter ($10^{-15}$ m). (6 points)

Solution:
Using the same argument to find the mean density of the neutron as for the star above we get:

$$\rho_{\text{neutron}} = \frac{m_n}{4\pi r_n^3/3} = \frac{1.7 \times 10^{-27} \, \text{kg}}{4 \times 3 \times (10^{-15} \, \text{m})^3/3} = 4 \times 10^{17} \, \text{kg/m}^3.$$

Note that the mean density of the NS is very close to the nuclear density! You should not be worried that the numbers don’t match exactly – this is an order of magnitude estimate! Being close to nuclear density means that the neutrons are almost touching each other inside the NS. This is the most dense configuration of matter we know of – anything denser than this would collapse into a black hole.

c) Calculate the mean distance between atoms in WD. Express your result in Ångstroms ($1$ Ångstrom = $10^{-10}$ meters). You may approximate the white dwarf to be made entirely of carbon atoms (12 times the mass of hydrogen). Compare with the typical sizes of atoms under normal conditions, $\sim 1$ Ångstrom. *Hint: We are looking for an approximate answer here.* (7 points)

Solution
To calculate the mean distance between atoms in WD, we need to ask for the volume which one carbon atom finds itself in inside the star. To do this, we need to know the number of Carbon atoms in the WD and the volume of the WD. The number can be obtained from the mass of the star ($M_{\text{WD}}$) and the mass of one Carbon atom ($M_C = 12m_p$, where $m_p$ is the mass of the proton).

$$N_C = \frac{M_{\text{WD}}}{M_C} = 2 \times 10^{30} \, \text{kg}/(12 \times 1.6 \times 10^{-27} \, \text{kg}) \approx 1/10 \times 10^{57} = 10^{56} \, \text{atoms}.$$

Note, that not every calculator will be able to deal with numbers this big, so adding powers of 10 by hand may be necessary! The volume per atom is then $(4\pi R_{\text{WD}}^3/3)/N_C =$
\[10^{21} \text{m}^3/(10^{56}) = 10^{-35} \text{ m}^3. \] Let us approximate each atom as occupying a cube of side \( L \) and thus volume \( V = L^3 \), so the mean distance between atoms is roughly \( L \) (ignoring a factor of 2):

\[ L = (10^{-35})^{1/3} \text{ m} \approx 2 \times 10^{-12} \text{ m} = 0.02 \text{ Å}. \]

This is 50 times smaller than the size of the atom under normal conditions; the atoms are tremendously squeezed in a white dwarf! In fact, the matter inside the white dwarf is not in atomic state – it is a sea of carbon nuclei and electrons, a plasma. However, when electrons are packed this close together, they start to exert the quantum mechanical “degeneracy” pressure, and this is what supports the white dwarf against collapse.

*The comparison with the size of ordinary atoms is worth 2 points. Any reasonable calculation gets full credit. No points off for missing factors of 2.*

d) We can estimate the rate of rotation of compact objects by knowing that they are the result of contraction of rotating main sequence stars. Each piece of gas in the star contracts in such a way that the product of its distance to the rotation axis times the velocity of rotation about the axis is a constant. This is known as conservation of “angular momentum,” and is the same phenomenon that causes a spinning ice skater to turn faster when she raises her hands. Consider a point on the equator of a main-sequence star with the radius of the Sun, and rotation period 30 days. Now imagine that this point contracted with the star and ends up on the surface of a WD or a NS. Find the rotation periods of the WD and NS. Express them in minutes and milliseconds, respectively. (*10 points*)

**Solution**

The conservation of angular momentum implies that the product of velocity of rotation of the point on the equator, \( v \) times the equatorial radius \( R \) is constant, or \( v \times R = \text{const} \). We can express the rotation velocity of a point on the equator in terms of the period of rotation, \( P \), or \( v = 2\pi R/P \). Substituting back, we get \( R^2/P = \text{const} \). Therefore, if the radius shrinks, the period would have to become shorter as well (the star spins up). Let’s denote the original period and radius with the subscript MS for main sequence, and final period and radius with subscript CO (for compact object). Then we have

\[ \frac{R_{\text{MS}}^2}{P_{\text{MS}}} = \text{const} = \frac{R_{\text{CO}}^2}{P_{\text{CO}}}. \]

Expressing the period of the compact object from this equation, we get:

\[ P_{\text{CO}} = \left( \frac{R_{\text{CO}}}{R_{\text{MS}}} \right)^2 P_{\text{MS}}. \]

Checking the sanity of the formula, we see that since \( r_{\text{CO}}/r_{\text{MS}} \ll 1 \) the period of rotation of the compact object will be much shorter, as we expect. Substituting numbers, \( R_{\text{WD}}/R_{\text{MS}} = R_{\text{Earth}}/R_{\odot} \approx 10^{-2} \), and \( R_{\text{NS}}/R_{\text{MS}} = 10\text{km}/R_{\odot} \approx 10^{-5} \). The period is then \( P_{\text{WD}} = (10^{-2})^2 \times 30\text{days} \approx 4 \text{ minutes} \), and for the neutron star, \( P_{\text{NS}} = (10^{-5})^2 \times 30\text{days} \approx 0.25 \text{ milliseconds} \). This number is actually near the limit of how
fast a neutron star can rotate (about 1 millisecond). From observations of radio pulsars we infer that neutron stars are born with periods from 1 to 20 milliseconds and then spin down with time. *Full credit for periods in the same ball park.*

### 3. Distance to a Supernova (25 points total)

a) The Crab Nebula, a supernova remnant, is roughly spherical, and is expanding. The angle it subtends on the sky increases at a rate of 0.23 arcseconds a year. Spectra taken of the glowing gases indicate, via the Doppler shift, that the gas is also expanding along the line of sight at 1200 km s$^{-1}$ relative to the center. From this, deduce an approximate distance to the Crab Nebula, in light years. *(8 points)*

**Solution:** We are told that the Crab Nebula is approximately spherical, and that it is expanding uniformly in all directions. In that case, the velocity of the outer shell will be uniform, 1200 km s$^{-1}$. This causes the diameter to increase at a rate of 0.23 arcseconds per year, corresponding to a rate of change of the distance from the center to the edge of half that, 0.12 arcseconds per year (as both “sides” of the shell are expanding away from the center). In one year, the outer shell moves a distance:

$$1200 \text{ km s}^{-1} \times 3 \times 10^7 \text{ s yr}^{-1} = 3.6 \times 10^{10} \text{ km.}$$

That corresponds to 0.12 arcseconds at the distance $d$ of the supernova remnant, so we can use the small-angle approximation:

$$0.12 \text{ arcsec} \times \frac{1 \text{ radian}}{200,000 \text{ arcsec}} = \frac{3.6 \times 10^{10} \text{ km}}{d}$$

Solving for $d$ gives

$$d = 3.6 \times 10^{10} \text{ km} \times \frac{2 \times 10^5}{0.12} = 6 \times 10^{16} \text{ km} \times \frac{1 \text{ ly}}{10^{13} \text{ km}} = 6000 \text{ ly.}$$

_Two points off for a factor-of-two error in the answer due to getting radii and diameters mixed up._

b) The angular size of the Crab nebula is currently $5 \pm 1.5$ arcminutes in diameter (the error reflects the fact that the nebula is not exactly spherical). Assuming that it has been expanding at a constant rate, calculate roughly the year at which the light from the initial supernova reached the Earth. Be sure to include the effects of the uncertainty in the angular size. Compare your result with the date at which the Chinese observed this supernova, AD 1054. Do these dates agree within the precision of your calculation? *(5 points)*

**Solution:** If we assume that the expansion rate has been constant throughout the history of the supernova, then it has been expanding for a time given by the ratio of its current size divided by the rate of expansion (a calculation just like that of the time it takes to travel a certain distance at a certain speed). Thus the time the supernova has been expanding is

$$\frac{(5 \pm 1.5) \text{ arcmin} \times 60 \text{ arcsec/arcmin}}{1/4 \text{ arcsec/yr}} = 1200 \pm 300 \text{ yr.}$$
We conclude that the supernova went off between around 500 and 1100 AD (i.e., between 900 and 1500 years ago). The observations of the supernova in 1054 CE are consistent with the latter date.

Of course, the supernova is at a distance of 6000 light years, and therefore we are seeing events there as they occurred 6000 years ago. It is the same 6000 year delay between different events in the history of the supernova. So the more correct way to phrase the above is that the light from the supernova explosion reached us between 500 and 1100 AD.

**Three points for the calculation of the time the supernova has been expanding. One point for calculation of the errorbar. One point for any cogent discussion of the relationship between the age of the supernova and the date 1054 AD. Give only one point for stating that the supernova went off 6000 years ago, because that is the light travel time to us.**

c) You are given the velocity of the gas above, and assume that the mass of the gas is twenty solar masses. Calculate the kinetic energy of the explosion; express your result in Joules. The star that exploded was probably an O star, with a main sequence luminosity $10^3$ times that of the Sun. Calculate how long the O star has to shine to generate as much energy as is currently present in the kinetic energy of the expanding gases. (7 points)

**Solution:** The kinetic energy of a mass $m$ moving at speed $v$ is $\frac{1}{2}mv^2$, so the total amount of energy is:

$$E = \frac{1}{2}mv^2 = \frac{1}{2} \times 20 M_\odot \times \frac{2 \times 10^{30} \text{kg}}{1 M_\odot} \times (1.2 \times 10^6 \text{m/s})^2$$

$$= 3 \times 10^{43} \text{Joules}.$$

where I was careful here to convert to MKS units. The luminosity of an O star, we are told, is $10^3$ times that of the Sun, or roughly $4 \times 10^{29}$ Joules/second. Thus the time the O star has to shine to generate that much energy is the ratio of the two numbers, or:

$$\text{Time} = \frac{3 \times 10^{43} \text{Joules}}{4 \times 10^{29} \text{Joules/sec}} = \frac{3}{4} \times 10^{14} \text{sec} \times \frac{1 \text{yr}}{3 \times 10^7 \text{sec}} = 2.5 \text{ million years}.$$

Note that this time is comparable to the lifetime of an O star!

**Four points for doing the energy calculation alone.**

d) Another way to estimate the date of the explosion is to study how fast the neutron star in the middle of Crab Nebula is slowing down. This neutron star is detected as a radio pulsar with period of $P = 33$ milliseconds. It is losing its rotational energy to a wind, and as a result is currently slowing down at the rate $r = 4.2 \times 10^{-13}$ seconds per second (the units of this rate are a bit funny – this is the number of seconds by which the period of the star is increasing every second). The characteristic age of a pulsar is given by the expression $P/(2r)$, where the factor of 2 accounts for the fact that the
pulsar was spinning down faster in the past. Calculate the characteristic age of the Crab pulsar and compare the date of the historical explosion to the date inferred from pulsar age determination. (5 points)

**Solution**
The estimated age based on the spindown of the Crab pulsar is:

\[
\text{age} = \frac{P}{2r} = \frac{0.033\text{s}}{2 \times (4.2 \times 10^{-13}\text{s/s})} = 3.96 \times 10^{10}\text{s}
\]

We can convert this to years (1yr \(\approx 3 \times 10^7\) s) to get 1,250 years for the age of the Crab pulsar. This implies that the Crab pulsar formed in 750 A.D., which is about three centuries earlier than when the supernova was actually observed. This means that the spindown age is not a very accurate estimator of the age; however, it can be used for rough estimates. The reason for the inaccuracy is that the rate of spin down is a function of time, and this has not been fully accounted for in the simple formula above.

4. **True or False?** (20 points)
Determine if the following statements are true or false, and give the reasoning to support your conclusion in a short paragraph.

a) If the Sun had been born as a high-mass star some 4.5 billion years ago, rather than as a low-mass star, the planet Jupiter would probably have Earth-like conditions today, while Earth would be hot like Venus. (7 points)

**Solution**
This statement is false. If the Sun had been born as a high-mass star 4.5 billion years ago, it would have exploded as a supernova a long time ago.

*2 sympathy points for missing the age problem, and carrying on an argument about a hotter star producing warmer temperatures on Earth and Jupiter*

b) Globular clusters generally contain lots of white dwarfs. (6 points)

**Solution**
This statement is true. Since globular clusters contain lots of old stars, and the end-state of low-mass stars are white dwarfs, one would expect globular clusters to have lots of them.

c) If a 3.5 \(M_\odot\) main-sequence star is orbiting a 2.5 \(M_\odot\) red giant, the red giant must have been more massive than 3\(M_\odot\) when it was a main-sequence star. (7 points)

**Solution**
This statement makes sense. The 2.5\(M_\odot\) red giant had to be more massive than its companion at some point in the past in order for it to be more advanced in its evolutionary state than its companion. This is because the main sequence lifetime directly depends on mass. Two stars that are in a binary likely formed at the same time, so if the more massive one now is less evolved than the less massive one, this could only be if the 2.5\(M_\odot\) star used to be more massive and has lost some mass after its main sequence evolution (most likely it transferred the mass to the companion, or lost it to a wind).
In these problems, a correct answer without explanation or with completely egregious explanation gets 2 points. Any hint of the right thinking in the explanation gets full credit.