Solutions to Homework #6, AST 203, Spring 2012

Due in class (i.e., by 4:20 pm), Thursday May 3rd (last lecture of the course)
General grading rules: One point off per question (e.g., 1a or 1b) for egregiously ignoring the admonition to write in full sentences. One point off per question for inappropriately high precision (which usually means more than 2 significant figures in this homework). No more than 2 total points per problem off for not writing in full sentences, and two points per problem for overly high precision. Three points off for each arithmetic or algebra error. Further calculations correctly done based on this erroneous value should be given full credit. However, if the resulting answer is completely ludicrous (e.g., 10^{-30} *seconds for the time to travel to the nearest star, 50 stars in the visible universe), and no mention is made that the value seems wrong, take three points off. One point off per question for not being explicit about the units, or for not expressing the final result in the units requested. Specific instructions for each problem take precedence over the above. In each question, one cannot get less than zero points, or more than the total number the question is worth.*

100 total points

1. Rocket ships (10 points)
Two spaceships, each measuring 300 m in its own rest frame, pass by each other traveling in opposite directions. Instruments on board spaceship A determine that the front of spaceship A requires 1 microsecond to traverse the full length of B. What is the relative velocity v of the two spaceships (in units of the speed of light)?
Answer: From the point of view of spaceship A, spaceship B is approaching it with the relative velocity v. Also, in the frame of A the length of spaceship B is contracted by the Lorentz factor $\sqrt{1-(v/c)^2}$. The time it takes for the contracted length of B to pass the front of spaceship A is then

$$\Delta t = \frac{L\sqrt{1-(v/c)^2}}{v}.$$

Now we need to solve this equation for the relative velocity v. Let's multiply both sides by v and divide by the speed of light.

$$(v/c)\Delta t = (L/c)\sqrt{1-(v/c)^2}$$

Squaring both sides and doing the arithmetic, we find:

$$(v/c)^2 = (L/(c\Delta t))^2(1-(v/c)^2),$$

or,

$$(v/c)^2 = \frac{(L/(c\Delta t))^2}{1+(L/c\Delta t)^2}$$

Taking the square root, the relative velocity of the two spacecraft is:

$$\frac{v}{c} = \frac{L/(c\Delta t)}{\sqrt{1+(L/(c\Delta t))^2}}.$$

To get the numerical result, we need to calculate the ratio $L/(c\Delta t)$. The numbers are convenient: $L = 300\text{m}$, $c\Delta t = 3 \times 10^8\text{m/s} * 10^{-6}\text{s} = 300\text{m}$, so $L/(c\Delta t) = 1$. The relative velocity is then $v/c = 1/\sqrt{2} \approx 0.7$, or $v = 0.7c$.

4 points off for missing the Lorentz length contraction factor. 2 more points off for then getting the relative velocity of c and not commenting it does not make sense.

2. A Hitchhiker's Challenge (30 points)

> "A full set of rules [of Brockian Ultra Cricket, as played in the higher dimensions] is so massively complicated that the only time they were all bound together in a single volume they underwent gravitational collapse and became a Black Hole."
> *Chapter 17 of Life, the Universe and Everything, the third volume of the Hitchhiker's Guide to the Galaxy series (1982, Douglas Adams)*

A quote like that above is crying out for a calculation. In this problem, we will answer Adams' challenge, and determine just how complicated these rules actually are.

An object will collapse into a black hole when its radius is equal to the radius of a black hole of the same mass; under these conditions, the escape speed at its surface is the speed of light (which is in fact the defining characteristic of a black hole!). We can rephrase the above to say that an object will collapse into a black hole when its *density* is equal to the density of a black hole of the same mass.

a. Derive an expression for the density of a black hole of mass M. Treat the volume of the black hole as the volume of a sphere of radius given by the Schwarzschild radius. As the mass of a black hole gets larger, does the density grow or shrink? (5 points)
Answer: The Schwarzschild Radius of a black hole of mass M is

$$R_{Sch} = \frac{2\,GM}{c^2}.$$

The volume of a sphere of this radius is just the familiar $\frac{4}{3}\pi R_{Sch}^3$. The density is the mass divided by the volume, giving:

$$\text{Density} = \frac{M}{\frac{4}{3}\pi\left(\frac{2\,GM}{c^2}\right)^3} = \frac{3\,c^6}{32\,\pi G^3 M^2}.$$

A messy expression! The more massive the black hole, the *smaller* the density. Thus there is a mass at which the black hole has the density of paper, which is what we are trying to figure out.
2 points for pointing out that the density decreases with a larger black hole mass. Full credit for stating $\pi \approx 3$, and canceling the two, or similar approximations.

It is worth pointing out that the actual material which makes up the black hole is *much* smaller than this, at *much* higher density. The material is, as best we understand, crushed to a point, of extent of order the Planck radius...

b. Determine the density of the paper making up the Cricket rule book, in units of kilograms per cubic meter. Standard paper has a surface density of 75 grams per square meter, and a thickness of 0.1 millimeters. (5 points)
Answer: The density is the mass per unit volume. If we can figure out the *volume* of a square meter of paper (whose mass we know, 75 grams), we can calculate its density. The volume of a piece of paper is its area times its thickness. The thickness is 0.1 millimeter, or 10^{-4} meters, and so the volume of a square meter of paper is $1\,\mathrm{meter}^2 \times 10^{-4}\,\mathrm{meter} = 10^{-4}\,\mathrm{meter}^3$. Therefore, the density of paper is:

$$\rho = \frac{7.5 \times 10^{-2}\,\mathrm{kg}}{10^{-4}\,\mathrm{meter}^3} = 7.5 \times 10^2\,\mathrm{kg/m}^3,$$

similar to (but slightly less than) the density of water (remember, paper is made of wood, and wood floats in water!).

c. Calculate the mass (in solar masses), and radius (in AU) of the black hole with density equal to that of paper. (10 points)

Answer: Here we equate the expression for density we found in part (a) with the density we calculated in part (b), and solve for mass. Let's first do it algebraically:

$$\rho = \frac{3\,c^6}{32\,\pi G^3 M^2}$$

$$M = \sqrt{\frac{3\,c^6}{32\,\pi G^3 \rho}}.$$

Now let's plug in numbers. This will be fun without a calculator:

$$M = \sqrt{\frac{3\,(3 \times 10^8\,\mathrm{m/s})^6}{32\,\pi (2/3 \times 10^{-10}\mathrm{m}^3\,\mathrm{s}^{-2}\,\mathrm{kg}^{-1})^3 \times 7.5 \times 10^2\,\mathrm{kg/m}^3}} \approx 3 \times 10^{38}\,\mathrm{kg},$$

where I made all the usual approximations of $\pi = 3$, $3^2 = 10$, and so on. We need to express this in solar masses, so we divide by 2×10^{30} kg, i.e., one solar mass, to get $1.5 \times 10^8\,M_\odot$ as our final answer. A black hole 150 million times the mass of the Sun has the same density as a piece of paper...

Surely this is science fiction! Well, yes, "The Hitchhiker's Guide to the Galaxy" is indeed science fiction, but do such incredibly massive black holes actually exist? Indeed they do: the cores of massive galaxies (including our own Milky Way) do contain such enormous black holes. Indeed, the most massive such black hole known to exist is in the core of a particularly luminous galaxy known as Messier 87, with a mass of 3 *billion* solar masses.

We still have to calculate the Schwarzschild radius of a black hole. We could plug into the formula for a Schwarzschild radius and calculate away, but I prefer a simpler approach. I know the Schwarzschild radius is proportional to the mass of a black hole, and I happen to remember (indeed, it is on our formulas sheet) that a one solar mass

black hole has a Schwarzschild radius of 3 kilometers. So a 150 million solar mass black hole has a Schwarzschild radius 150 million times larger, or 4.5×10^8 kilometers. We are asked to express this in terms of AU; one AU is 1.5×10^8 kilometers, so the Schwarzschild radius of such a black hole is 3 AU.
Six points for the calculation of only the radius or the mass, but not both. Calculation of the Schwarzschild radius need not scale off the Sun, as was done here...

d. How many pages long is the Brockian Ultra Cricket rule book? Assume the pages are standard size ($8.5'' \times 11''$). For calculational simplicity, treat the book as spherical (a common approximation in this kind of problem). What if the rule book were even longer than you have just calculated? Would it still collapse into a black hole? (10 points)
Answer: We know the entire mass of the black hole. If we can calculate the mass of a single piece of paper, the ratio of the two gives the total number of pages. So let's calculate the mass of a single piece of paper. We know that a square meter of paper has a mass of 75 grams. How many square meters is a standard-size sheet? One inch is 2.5 centimeters $= 2.5 \times 10^{-2}$ meters. So $8.5 \times 11\,\mathrm{inch}^2 \approx 100\,\mathrm{inch}^2 \approx 6 \times 10^{-2}\,\mathrm{meter}^2$. Thus the mass is:

$$\text{Mass of a piece of paper} = 7.5 \times 10^{-2}\,\mathrm{kg/meter}^2 \times 6 \times 10^{-2}\,\mathrm{meter}^2 \approx 5 \times 10^{-3}\,\mathrm{kg}.$$

That is, a piece of paper weighs about 5 grams. We divide this into the mass we calculated above:

$$\text{Number of sheets of paper} = \frac{\text{Mass of rule book}}{\text{Mass per page}} = \frac{3 \times 10^{38}\,\mathrm{kg}}{5 \times 10^{-3}\,\mathrm{kg/page}} = 6 \times 10^{40}\,\text{pages}.$$

(Strictly speaking, if the rule book is printed on both sides of the page, we should multiply the above result by a factor of two.) That is one seriously long set of rules!

Finally, note that because the density of a more massive black hole is smaller, as we saw in part (a), the above mass and number of pages of the Brockian Ultra Cricket rule book is really just a lower limit. That is, if the rule book were even larger than what we've just calculated, it would still collapse into a black hole.
6 points for the calculation of the number of pages (and there is no need to include the factor of two!), and four points for answering the question about what would happen if the book were even longer.

3. Galaxy Rotation Curves and Dark Matter (*35 points*)

We will examine galaxy rotation curves and show that they imply the existence of dark matter.

a) Recall that the orbital period P is given by $P^2 = 4\pi^2 a^3/GM$. Write down an expression that relates the orbital period and the orbital velocity for a circular orbit, and then write down an expression that relates the orbital velocity with the mass enclosed within R. (*Hint: we've done this several times before*) (*5 points*)
Solution:

The circumference of a circle of radius a is $2\pi a$, so the orbital speed is the circumference divided by period:

$$v = \frac{2\pi a}{P} = \frac{2\pi a}{\sqrt{4\pi^2 a^3/GM}} = \sqrt{\frac{GM}{a}} \tag{1}$$

2 points for writing $2\pi a/P$ and going no further.

b) The Sun is 8,000 parsecs from the center of the Milky Way, and its orbital velocity is 220 km/s. Use your expression from a) to determine roughly how much mass is contained in a sphere around the center of the Milky Way with a radius equal to 8,000 parsecs? (*5 points*)
Solution:
According to the Birkoff's theorem, the orbit about a mass distributed within a sphere is the same as if the mass is all concentrated in the center of the sphere. So, we can use the velocity formula derived in a), and invert it to obtain the mass enclosed by an orbit:

$$M(a) = \frac{a}{G}v^2 = \frac{8 \times 10^3 \mathrm{pc}}{6.6 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}} (220 \mathrm{km/s})^2 \tag{2}$$

Note, that velocity and semimajor axis are not in SI units, so we need to convert, using 1 pc = 3.2 ly = 3×10^{16} m, and 220 km/s = 2.2×10^5 m/s. The enclosed mass at 8kpc is then: $M(8\mathrm{kpc}) \approx 2 \times 10^{41}\mathrm{kg} = 10^{11} M_\odot$.

c) Assume that the Milky Way is made up of only luminous matter (stars) and that the Sun is at the edge of the galaxy (not quite true, but close). What would you predict the orbital velocity to be for a star 30,000 parsecs from the center? 100,000 parsecs? (*10 points*)
Solution:
If the mass enclosed by the orbit stays at $10^{11} M_\odot$ as the radius increases, which follows from the fact that the Sun is at the edge of the luminous galaxy, then at different radii the velocity given by equation (1) will decrease with square root of the distance. At 30,000 pc, it is $(220\mathrm{km/s}) \times \sqrt{8000\mathrm{pc}/30000\mathrm{pc}} \approx 110$ km/s. At 100,000pc, $v = (220\mathrm{km/s}) \times \sqrt{8000/100000} \approx 60$ km/s.

d) *Observations* show that galaxy rotation curves are flat: stars move at the same orbital velocity no matter how far they are from the center. How much mass is *actually* contained within a sphere of radius 30,000 parsecs? 100,000 parsecs? Take the orbital velocity at these radii to be the same as the orbital velocity of the Sun. (*10 points*)
Solution:

Let's look again at equation (1):

$$M(a) = \frac{a}{G}v^2, \tag{3}$$

which says that if the orbital velocity stays the same, the mass enclosed will increase linearly with a as the radius of the orbit grows. We already calculated the mass

enclosed by 8,000 pc orbit in part b), and we got $10^{11}M_\odot = 2\times10^{41}$kg. So, at 30,000 pc, the mass enclosed will be 30,000pc/8,000pc times larger, or $3.8\times10^{11}M_\odot = 7\times10^{41}$kg. At 100,000pc, the mass enclosed will 100,000pc /8,000pc times larger, or $13\times10^{11}M_\odot \approx 3\times10^{42}$kg.
(*roundings may differ a bit for different people here.*)

e) What do you conclude from all of this about the contents of our galaxy? (*5 points*)
Solution: We see that the mass of the gravitating matter is increasing linearly with radius, and exceeds by more than factor of 10 the mass of the luminous matter (e.g. stars and gas). We thus infer that the outer halo of the galaxy is dominated by invisible dark matter.

4. Alternative Universe (*25 points*)

Imagine Alan Sandage was right, and the Universe had a Hubble Constant of 50 km/s/Mpc. Calculate each of the following, and compare with the values for the real Universe:

a) You observe a galaxy with the hydrogen 656.3 nm line redshifted to a value of 926 nm. Calculate the redshift for this galaxy, and determine the distance to the galaxy in the "Alan" Universe and the real Universe. (*5 points*)
Solution:
The redshift of the galaxy: $z = (\lambda_{\text{observed}} - \lambda_{\text{emitted}})/\lambda_{\text{emitted}} =$
$(926\text{nm} - 656.3\text{nm})/656.3\text{nm} \approx 0.4$.

We know that $z = v/c$, or using the Hubble's law:

$$v = cz = H_0 d \tag{4}$$

so

$$d = \frac{cz}{H_0} = \frac{0.4\times3\times10^5}{50} = 2400\text{Mpc} \tag{5}$$

In the real universe the distance will be a factor of $\frac{50}{70} = 0.7$ smaller, so $d = 1700$ Mpc. The value of the Hubble constant does not affect the measured redshift z, so that's still 0.4.

b) Calculate the age of the universe in the "Alan" Universe and compare with the real value. (*5 points*)
Solution:
We saw in class that the time for all galaxies to be in the same place can be calculated using: $vt = d$ and $v = H_0 d$. We find the simple relationship that $t = \frac{1}{H_0}$. For the real universe, the answer is 14 billion years, so for the Alan universe, it is a factor of $\frac{70}{50} = 1.4$ older or 19.6 billion years old. In fact, one of Sandage's reason for preferring a low Hubble constant was to have a universe older than the oldest stars.

Here is the calculation of the age of the universe in the Alan Universe:

$$t = \frac{1}{H_0} = \frac{1}{50\text{kms}^{-1}\text{Mpc}^{-1}} \times \frac{3\times10^{19}\text{km}}{1\text{Mpc}} \times \frac{1\text{yr}}{3\times10^7} \approx 2\times10^{10}\text{yr} \tag{6}$$

c) How many galaxies are there in the observable Universe? Here you can assume that the average distance between galaxies is comparable to the distance between the Milky Way and Andromeda (2.5 mega light-years). Do this calculation for both Universes. (*10 points*)
Solution:
Let us assume that the average distance between galaxies is around $r = 3$ million light years. The observable universe has a radius of $R = 20$ billion light years from part 2. Thus we want to know how many spheres with volume $\sim 4r^3 \approx 10^{20}$ cubic light years fit in the volume of the observable universe, which is $4R^3 \approx 3 \times 10^{31}$ cubic light years. Taking 3×10^{31} divided by 10^{20} is 3×10^{11} galaxies.

In the real universe, the size is 14 billion years, and the volume is thus $(\frac{5}{7})^3$ or 40% smaller. The volume is thus 10^{31} cubic light years, and so the number of galaxies in the observable universe is just about 10^{11} galaxies.

d) Finally, using your answer to (c), calculate the total number of stars in the observable universe. Here you can assume that most galaxies are comparable to the Milky Way. (*5 points*)
Solution:
If we take the average number of stars in a galaxy to be 10^{10} stars, then the number of stars in the Alan universe is 3×10^{21} stars, or just 10^{21} stars in the real universe. (*Full credit for using* 2×10^{11} *stars/galaxy and corresponding correct calculation*)