Solutions for homework #5, AST 203, Spring 2012

Due in class (i.e., by 4:20 pm), Thursday April 19

General grading rules: One point off per question (e.g., 1a or 1b) for egregiously ignoring the admonition to write in full sentences. One point off per question for inappropriately high precision (which usually means more than 2 significant figures in this homework). No more than 2 total points per problem off for not writing in full sentences, and two points per problem for overly high precision. Three points off for each arithmetic or algebra error. Further calculations correctly done based on this erroneous value should be given full credit. However, if the resulting answer is completely ludicrous (e.g., 10^{-30} seconds for the time to travel to the nearest star, 50 stars in the visible universe), and no mention is made that the value seems wrong, take three points off. One point off per question for not being explicit about the units, or for not expressing the final result in the units requested. Specific instructions for each problem take precedence over the above. In each question, one cannot get less than zero points, or more than the total number the question is worth.

100 total points

1. Time Dilation (45 points)

In this problem, we will explore the nature of the function: $y = \sqrt{1 - \frac{v^2}{c^2}}$. This quantity, sometimes called the "Lorentz Factor" is the factor in Special Relativity by which an astronaut moving by at speed v ages. That is, I age 1 year while I observe that the moving astronaut ages y years.

We start by exploring the behavior of this function for small values of v, and those close to the speed of light. To do this, we'll need to develop a few mathematical tools. If we define $x = v/c$, the Lorentz Factor can be written as $y = \sqrt{1 - x^2}$.

a. (10 points) For very small velocities, $v \ll c$ and we expect y to be very close to (but slightly less than) 1. Thus we write $y = 1 - \epsilon$, where $\epsilon \ll 1$. Our exercise will be to determine ϵ. Solve the equation above for ϵ in terms of x: start by squaring both sides of the equation, and then recognize that if ϵ is small, ϵ^2 is tiny, and additive terms involving ϵ^2 can be neglected. With the value of ϵ in terms of x in hand, now express the Lorentz Factor y in terms of v and c, for $v \ll c$.

Answer: When $v \ll c$, then $v/c \ll 1$, and y will be very close to unity. We can do an approximation as follows. We know that: $(1-\epsilon)^2 = 1 - 2\epsilon + \epsilon^2$. Suppose that $\epsilon \ll 1$. In this case, ϵ^2 is an even smaller quantity, and we will make the approximation that it is negligible. Thus $(1-\epsilon)^2 = 1 - 2\epsilon$, to a very good approximation. We can turn this around, and write:

$$\sqrt{1 - 2\epsilon} = 1 - \epsilon.$$

Now,

$$y = \sqrt{1 - \left(\frac{v}{c}\right)^2} = \sqrt{1 - 2 \times \frac{1}{2}\left(\frac{v}{c}\right)^2},$$

just the form we have above, for $\epsilon = 1/2(v/c)^2$. Thus if $v \ll c$, then $\epsilon \ll 1$, and

$$y = \sqrt{1 - v^2/c^2} = 1 - \frac{1}{2}\frac{v^2}{c^2},$$

to a very good approximation.

Incidentally, this is an example of a general rule, which can be demonstrated with something called a *Taylor* series (which one learns about in calculus), namely:

$$(1+\epsilon)^n \approx 1 + n\epsilon,$$

whenever $|\epsilon| \ll 1$.

Given the hints given in the problem, most students will simply get this right. Two points off for never expressing the final result in terms of v and c (i.e., just keeping the result in terms of x). Five points off for each algebraic error. Full credit for using the tools of calculus, and phrasing the answer in terms of a Taylor expansion, if they explain this clearly.

b. (10 points) Now let's take the opposite limit, namely speeds very close to the speed of light. This time, we'll write $x = v/c = 1 - \alpha$, where now $\alpha \ll 1$. Plug into the equation for the Lorentz Factor; similar to part a, if α is small, then α^2 is really tiny and can be neglected. Thus write an expression for the Lorentz Factor in this case.

Answer: Now let's take the opposite limit, where v is very close to c. In particular, let's write $v/c = 1 - \alpha$, where $\alpha \ll 1$. Note that $v^2/c^2 = (1-\alpha)^2 = 1 - 2\alpha$, to a good approximation. Thus:

$$y = \sqrt{1 - v^2/c^2} = \sqrt{1 - (1 - 2\alpha)} = \sqrt{2\alpha} = \sqrt{2(1 - v/c)}.$$

Same grading policy as in (a). Four points off for keeping terms of order $(v/c)^2$.

c. (5 points) Now we're ready to plug in some numbers. Draw a graph of the Lorentz Factor as a function of velocity, where the x-axis ranges from 0 to the speed of light c, and the y-axis ranges from 0 to 1. Plug in many values of v and calculate the value of the Lorentz Factor, and plot them up. Describe the shape of this function in words. Next, plot on the same graph the two approximations you calculated, for small velocities and large velocities, in parts (a) and (b). Over what range of velocities does each do a decent job of approximating the Lorentz value? That is, over what range does each approximation give answers within 10% of the correct value?

Answer: The graph is shown on the next page, showing both the exact formula for the Lorentz contraction, and the two approximations we've just worked out. The approximations work impressively well. The overall shape of the curve is that of a piece of a circle, extending from unity at small speeds, to zero at the speed of light. It is within 10% of the approximation worked out in part (a) for speeds less than $0.76c$, and with the approximation worked out in part (b) for speeds above $0.65c$.

3 points for the figure; take 1 point off if the three curves are not clearly labelled, 1 point off if only one approximation is shown, and 2 points off if neither approximation is shown. One point for any reasonable (1-2 sentence) description of the curves, and two points for any reasonable statement about the range of velocities in which the approximations seem to work. Indeed, even though we ask for the point at which the

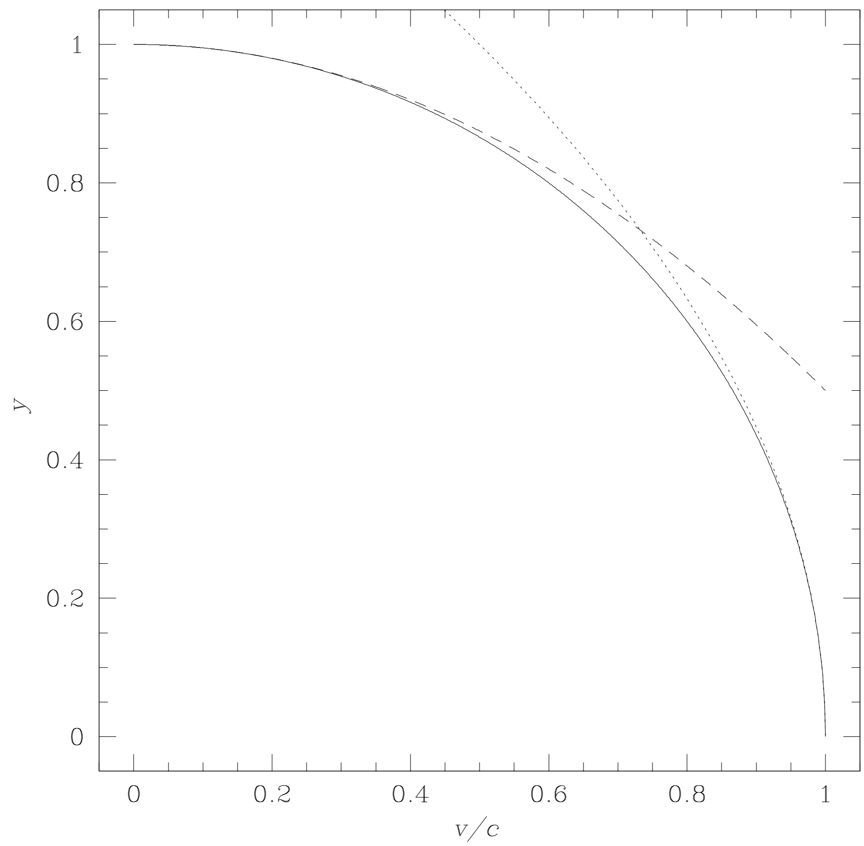

Figure 1: **Figure for Problem 1c.** The solid line is the Lorentz contraction formula. The dotted and dashed lines are the approximations worked out in parts (b) and (a), for high v/c and low v/c, respectively.

approximations are wrong by 10%, any reasonable qualitative statement about the range of validity of the curves gets full credit. Full credit for curves drawn using graphing software, or done by hand. Give no more than 2 points for hand-drawn curves which have little relationship to the true shape of the curves (i.e., where they didn't calculate any values directly).

d. (5 points) What is the value of y for $v = 0.001c$, $v = 0.6c$, $v = 0.8c$, $v = 0.99995c$, $v = 0.9999995c$? Use the approximations you've just developed, as appropriate. If you find yourself rounding off and saying that $y = 1$ or $y = 0$, you've rounded too much!

Answer: Here we have to calculate the Lorentz factor for various values of v. For $v = 0.001c$, we'll use the approximation of part (a), and for the two values of v very close to the speed of light, we'll use the approximation of part (b) (that's what they are for!). For $v = 0.6c$ and $0.8c$, we'll do the full calculation, and also show the results for the two approximations; we'll see that both do a pretty good job!

$v = 0.001c$, $y = 1 - 1/2(v/c)^2 = 1 - 1/2 \times 10^{-6} = 1 - 0.0000005 = 0.9999995$.

$v = 0.6c$, $y = \sqrt{1 - 0.6^2} = \sqrt{1 - 0.36} = \sqrt{0.64} = 0.8$. For the small-$v$ approximation, we get $y = 1 - 1/2(0.6^2) = 1 - 1/2 \times 0.36 = 0.82$, pretty close! For the large-v approximation, we get $y = \sqrt{2(1 - 0.6)} = \sqrt{0.8} \approx 0.9$; not as close to the right answer.

$v = 0.8c$, $y = \sqrt{1 - 0.8^2} = \sqrt{1 - 0.64} = \sqrt{0.36} = 0.6$. For the small-$v$ approximation, we get $y = 1 - 1/2(0.8^2) = 1 - 1/2 \times 0.64 = 0.68$, a bit high. For the large-v approximation, we get $y = \sqrt{2(1 - 0.8)} = \sqrt{0.4} \approx 0.63$; pretty good!

$v = 0.99995c$, $y = \sqrt{2(1 - 0.99995)} = \sqrt{2 \times 5 \times 10^{-5}} = \sqrt{10^{-4}} = 10^{-2}$.

$v = 0.9999995c$, $y = \sqrt{2(1 - 0.9999995)} = \sqrt{2 \times 5 \times 10^{-7}} = \sqrt{10^{-6}} = 10^{-3}$.

2 points for each calculation. No point for a given part if the student simply rounds to $y = 1$ or $y = 0$. In the first and last one, their calculator will be challenged if they try to calculate it "exactly"; it is actually more accurate to do it using the approximation. Take off one point in the first, fourth and fifth points if they do not use the relevant approximation, and attempt to do it exactly. Full credit for $0.6c$ and $0.8c$ for either approximation used.

e. (15 points) A muon is a particle very much like an electron, only with more mass. Unlike electrons, muons are unstable; they have a half-life of only 2.2×10^{-6} seconds, after which they decay into an electron, a muon neutrino, and an anti-electron neutrino. Very fast-moving muons are produced in the upper atmosphere of the Earth (i.e., 100 km above the Earth's surface) in collisions with high-energy cosmic rays (produced ultimately from distant supernova explosions), and come whizzing down to be detected on the surface. A typical atmospheric muon is moving at 0.9999995 the speed of light; i.e., $(1 - 5 \times 10^{-7})c$. Taking into account time dilation, how far could such a muon travel before decaying (i.e., before one half-life is over)? Now do the same calculation, not taking into account time dilation. Discuss: does the fact that we observe muons that were produced 100 km away give any support to Einstein's prediction of time dilation?

Answer: In part (d), we calculated the Lorentz contraction factor for this value of the speed; it is 10^{-3}. That is, time for the muon progresses 1000 times slower than it does for us. Thus the muon now has a lifetime 1000 times *longer*, i.e., 2.2×10^{-3} seconds. How far does it go at almost the speed of light? Well, light travels at 1 foot per nanosecond, and its lifetime is 2.2 million nanoseconds, so it goes 2.2 million feet, or about 700 kilometers.

If it weren't for the effects of time dilation, it would have its usual lifetime of 2.2 microseconds, and would therefore be able to travel 1/1000 as far, i.e., about 700 meters.

Thus the observation of muons which have travelled 100 kilometers is a direct demonstration of time dilation at work; it simply would be impossible to observe cosmic ray muons at the Earth's surface if Einstein was not correct...

8 points for the calculation of the distance the muon could travel before decaying. Give only 2 points if they multiply, rather than divide, by the factor 10^{-3}*. 4 points for the calculation ignoring time dilation. Full credit for any correct units. Full credit for results consistent with part (d), even if the former is incorrect. 3 points for any reasonable discussion (even just a sentence or two) explaining what this says about the veracity of Lorentz contraction. Only give full credit if the statement is consistent with the numbers. That is, if the numbers are incorrect, and state that the muons cannot travel 100 km before decaying, then the statement should definitely say that there is something wrong! The problem erroneously called time dilation as time contraction in the last sentence. Don't penalize for strange*

2. Antimatter! (10 points)
Two 10-metric-ton trucks (each with mass = 10^4 kilograms), each going 100 kilometers per hour in opposite directions have a head-on collision. Each truck has a kinetic energy of $\frac{1}{2}mv^2$ according to Isaac Newton. Since the velocity is small relative to the speed of light, Newton's formula is quite accurate. When the trucks collide, the energy of the resulting explosion is equal to the total original kinetic energy of the two trucks. Now suppose that a 10-metric-ton truck made out of matter collides with a 10-metric-ton truck made out of anti-matter, so that the two trucks annihilate, causing an explosion in which the total mass of the two trucks is converted into energy according to Einstein's formula $E = mc^2$. Assume that the matter and anti-matter trucks hit at such a low velocity that their kinetic energy can be ignored. Calculate the ratio of the energy of collision of the matter and anti-matter trucks to that of the two normal trucks in the head-on collision described above. If you see a matter and an anti-matter truck about to hit each other, should you try to get far away?

Solution: First, anti-matter has mass just like ordinary matter; it doesn't have "negative mass" in any sense of the term. So it feels gravity, and is equivalent to energy in the usual $E = mc^2$ sense. What makes it *anti*-matter is just the sense described in the problem, namely the fact that when it combines with ordinary matter, it annihilates completely. Nasty stuff to have around...

Let's start this problem algebraically, and plug in numbers in the end. We have two trucks, each of mass m, each traveling at speed v. The kinetic energy of each is $1/2mv^2$, so the two of them together have a total kinetic energy of mv^2.

Now, for the matter and anti-matter trucks, they each have a rest-mass energy of mc^2 (same m as above!), and thus the two together have a total energy of $2\,mc^2$, all of which will be released upon collision. Wow! The problem asks: what is the ratio of these two energies? That's straightforward:

$$\text{Ratio} = \frac{2\,mc^2}{mv^2} = 2\left(\frac{c}{v}\right)^2.$$

Note that the mass drops out of this problem. OK, let's plug in numbers:

$$\text{Ratio} = 2\left(\frac{3\times 10^5\,\text{km/sec}}{100\,\text{km/hour}\times 1\,\text{hour}/3600\,\text{sec}}\right)^2 = 2\times(3000\times 3600)^2$$

$$\approx 2\times 10^{14}.$$

(In the last step, I approximated $3\times3.6 \approx 10$). That is, the collision of the matter-antimatter trucks is 200 trillion times more energetic than the head-on collision of two ordinary trucks, each traveling at highway speeds. You definitely had better run if you see such a collision about to happen! Luckily, there is not much anti-matter around (if there were some, it would instantly annihilate with whatever ordinary matter is around, thereby destroying it). It is actually a real cosmic mystery why it is that the universe did not create equal amounts of matter and anti-matter in the Big Bang. It is a good thing too: again, if equal amounts were created, it would all annihilate, leaving a universe of lots of energy (in the form of photons) and no material objects. A rather boring place...

Note that we didn't need to know the masses of the trucks to calculate the ratio. But for completeness, let's just work out how much energy this matter-antimatter collision actually releases: $2\,mc^2 = 2\times10^4\,\mathrm{kg}\times(3\times10^8\,\mathrm{m/sec})^2 = 2\times10^{21}$ Joules. Note that our Sun produces 200,000 times this much energy every *second*. That is, every second, the Sun turns roughly 4 million tons of matter into pure energy (via the process of nuclear fusion, not matter-antimatter annihilation).

1 points for any reasonable answer to the question, should you try to get far away? The problem asks for the ratio of the two energies; if this is not calculated, then up to 4 points for calculating the kinetic energy of the trucks, or the rest-mass energy and 6 points for doing both.

3. What's at the center of our Galaxy? (15 points)

You observe a star on a circular orbit around the center of our Galaxy. The radius of the orbit is 20 light-days. From the Doppler shift of its absorption lines you determine that its orbital speed is 1,000 km/s. Based on your measurement, what would be your estimate for the mass of the object that the star is orbiting? What kind of an object do you think this is? Give arguments to support your conclusion. A movie of orbits of stars observed near Galactic center can be seen at http://tinyurl.com/5ck5rp for your entertainment.

Solution:

The mass enclosed by a circular orbit can be determined knowing the radius of the orbit and the orbital velocity:

$$M = \frac{rv^2}{G}.$$

We derived this expression in class when we talked about the galactic rotation curve, and in the form expressing v through M and r in homework 2. This is a simple consequence of the Kepler's third law, or of Newton's law of gravitation: $P^2 = 4\pi^2 a^3/GM$. Remember, that $v = 2\pi a/P$. Expressing unknown P through $P = 2\pi a/v$ and substituting this into the third kepler law, we get: $v^2 = GM/r$, or the expression anbove. To use this expression, we need to convert the radius and the speed of the star into meters and meters per second. The speed we find to be 1.0×10^6m/s. The radius is a bit more work, since it is given as 20 light-days. There should be 365 light-days in a light-year, so we can convert and find that the orbital radius is 5.5×10^{-2} light-years $= 5.5\times10^{-2}\times3\times10^8m/s\times3\times10^7s= 5.2\times10^{14}$m. The mass enclosed by the orbit is, therefore, $5.2\times10^{14}\mathrm{m}\times(1.0\times10^6\mathrm{m/s})^2/(2/3\times10^{-10}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}) = 7.8\times10^{36}$ kg. This is about 4 million solar masses!

What kind of an object can this be? Let's proceed with the process of elimination. Can this be a star? No, because we know that the maximum mass of a star is 150 solar masses. Can this be many stars, perhaps a cluster of stars? Globular clusters can have up to a million stars in them. This is a bit on the low side, given that a typical star is 0.5 solar masses. But, even if it were a globular cluster, it would be way too compact – we want it to fit inside the orbit of 5.2×10^{14}m, which is roughly 0.01 of a parsec. A typical globular cluster would be at least several parsecs in radius (typically, 10pc). In fact, we know of only one type of object that can accommodate such an enormous mass confined to such a small volume – a black hole! The black hole in the center of our Galaxy is actually quite tiny compared to other galaxies – black holes up to a few billion solar masses are known to exist in the centers of many distant galaxies.

7 points for the calculation, 3 points for the explanation.

4. The Hubble Constant (30 points total)
In this problem, you will use data from the Sloan Digital Sky Survey to calculate the Hubble Constant. The attached figure shows the spectra of a star and four galaxies, as measured by this project; all these objects are included in the wall mosaic. For each of them, we indicate the measured brightness, in units of Joules per square meter per second. Assume that each of them has the same luminosity as that of the Milky Way (10^{11} times the luminosity of the Sun, or 4×10^{37} J/s). This is a problem in which the use of a calculator is appropriate.

a. (10 points) Determine the distance to each of the four galaxies, using the inverse-square law relation between brightness and luminosity. Express your answers both in meters and in Megaparsecs (1 Mpc = 3.1×10^{22} meters), and give two significant figures.
Answer: The relationship between luminosity, distance, and brightness is given by the inverse-square law, namely:

$$\text{Brightness} = \frac{\text{Luminosity}}{4\,\pi\,\text{Distance}^2}$$

Here we are given the luminosity of each galaxy (the four are the same, namely 4×10^{37} Joules/second), and the brightness, in units of Joules/meters^2/second. Solving for the distance gives:

$$\text{Distance} = \left(\frac{\text{Luminosity}}{4\,\pi\,\text{Brightness}}\right)^{1/2}$$

Note that the numbers will come out in meters, which is just fine. It is straightforward in each case to divide by 3.1×10^{22} meters/Mpc, to get the distance in Megaparsecs. When we do this for the four galaxies, we find:
Galaxy 1, Distance = 6.5×10^{24} meters = 210 Mpc
Galaxy 2, Distance = 8.4×10^{24} meters = 270 Mpc
Galaxy 3, Distance = 1.1×10^{25} meters = 360 Mpc
Galaxy 4, Distance = 1.5×10^{25} meters = 490 Mpc
Four points off for using the wrong expression for distance, but calculating consistently for all four galaxies. Two points off for each galaxy for which calculations are not done. Three points off for expressing results only in meters, or only in Mpc.

b. (10 points) Measure the redshift of each galaxy. That is, calculate the fractional change in wavelength of the Calcium lines. *Hint: The tricky thing here is to make sure you're identifying the right lines as Calcium. In each case, they are a close pair; for Galaxy #2, they are the prominent absorption dips between 4100 and 4200 Ångstroms. Measure the redshift by measuring both of the lines in each galaxy (and in each case, the two lines should* *give the same redshift, of course!). In this problem, please give your final redshift to two significant figures. Do the intermediate steps of the calculation without rounding; rounding too early can result in errors.*

Answer: The following is a table of measured wavelengths for each of the two lines in each of the galaxies, the corresponding redshift from each of the lines, and the average redshift. The redshift is given by $z = (\lambda - \lambda_0)/\lambda_0$, where $\lambda_0 = 3935$Å and 3970Å for the two lines.

Galaxy	λ First line	λ Second Line	Redshift First Line	Redshift Second Line	Average Redshift
1	4100Å	4135Å	0.042	0.041	0.042
2	4145	4185	0.053	0.054	0.053
3	4215	4255	0.071	0.072	0.071
4	4318	4360	0.097	0.098	0.098

Full credit for anything reasonable, even if they calculate redshifts based on only one line. Two points off for getting the two lines mixed up. Six points off for completely mis-identifying the lines. Two points off for each galaxy in which the calculation is not done. No points off for giving results in velocity (km/s) rather than redshift.

c. (10 points) Given the redshifts, calculate the velocity of recession for each galaxy in kilometers per second, and in each case use the distances to estimate the Hubble Constant, in units of kilometers per second per Megaparsec. You will not get identical results from each of the galaxies, due to measurement uncertainties (but they should all be in the same ballpark), so average the results of the four galaxies to get your final answer.

Answer: The redshift is equal to the velocity of recession divided by the speed of light. So we can calculate the velocity of recession as the redshift times the speed of light (we'll use $c = 3 \times 10^5$ km/sec, as this gives the units for the Hubble Constant we want to find).

The Hubble constant is given by the ratio of the velocity of each object to its distance. So we make another table:

Galaxy	Redshift	Velocity (km/s)	Distance (Mpc)	H = V/D (km/s/Mpc)
1	0.042	12600	210	60
2	0.053	15900	270	59
3	0.071	21300	360	59
4	0.098	29400	490	59

The four galaxies give consistent values of the Hubble Constant, at about 60 km/s/Mpc. Not identical to the modern value of 70 km/s/Mpc, but close.

That seemed quite straightforward; so why is there such controversy over the exact value of the Hubble Constant? The difficult point is getting an independent measurement of the luminosity of each galaxy. The problem stated that each of the galaxies has the same luminosity of the Milky Way. That is only approximately true; the numbers were adjusted somewhat to make this come out with a reasonable value for H.

Two points off for not calculating the velocity in each case. Two points off per galaxy for not doing the calculation of the Hubble Constant for it. Two points off for result in different units. Two points off for not averaging, or making some statement about agreement between the different galaxies.