Homework #5, AST 203, Spring 2012

Due in class (i.e., by 4:20 pm), Thursday April 19

- To receive full credit, you must give the correct answer *and* show that you understand it. This requires writing your explanations in full, complete English sentences, clearly labeling all figures and graphs, showing us how you did the arithmetic, and being explicit about the units of all numbers given. All relevant mathematical variables should be explicitly defined. And please use your best handwriting; if we can't read it, we can't give you credit for it! Please staple together the sheets of paper you hand in.

- Most of the calculations in this course involve numbers that are only approximately known. The result of such a calculation should reflect this imprecision. In particular, it is *wrong* to simply write down all the digits that your calculator spits out. Your final answer should have the same number of significant figures as the least precise number going into your calculation. In many (but not all!) cases, it's best to do the problems without a calculator.

- Feel free to work with your classmates on this homework, but your write-up and wording should be your own. Answer all questions.

100 total points

1. Time Dilation (45 points)

In this problem, we will explore the nature of the function: $y = \sqrt{1 - \frac{v^2}{c^2}}$. This quantity, sometimes called the "Lorentz Factor" is the factor in Special Relativity by which an astronaut moving by at speed v ages. That is, I age 1 year while I observe that the moving astronaut ages y years.

We start by exploring the behavior of this function for small values of v, and those close to the speed of light. To do this, we'll need to develop a few mathematical tools. If we define $x = v/c$, the Lorentz Factor can be written as $y = \sqrt{1 - x^2}$.

a. (10 points) For very small velocities, $v \ll c$, we expect y to be very close to (but slightly less than) 1. Thus we write $y = 1 - \epsilon$, where $\epsilon \ll 1$. Our exercise will be to determine ϵ. Solve the equation above for ϵ in terms of x: start by squaring both sides of the equation, and then recognize that if ϵ is small, ϵ^2 is tiny, and additive terms involving ϵ^2 can be neglected. With the value of ϵ in terms of x in hand, now express the Lorentz Factor y in terms of v and c, for $v \ll c$.

b. (10 points) Now let's take the opposite limit, namely speeds very close to the speed of light. This time, we'll write $x = v/c = 1 - \alpha$, where now $\alpha \ll 1$. Plug into the equation for the Lorentz Factor; similar to part a, if α is small, then α^2 is really tiny and can be neglected. Thus write an expression for the Lorentz Factor in this case.

c. (5 points) Now we're ready to plug in some numbers. Draw a graph of the Lorentz Factor as a function of velocity, where the x-axis ranges from 0 to the speed of light c, and the y-axis ranges from 0 to 1. Plug in many values of v and calculate the value of the Lorentz Factor, and plot them up. Describe the shape of this function in words. Next, plot on the same graph the two approximations you calculated, for small velocities and large velocities, in parts (a) and (b). Over what range of velocities does each do a decent job of approximating the Lorentz value? That is, over what range does each approximation give answers within 10% of the correct value?

d. (5 points) What is the value of y for $v = 0.001c$, $v = 0.6c$, $v = 0.8c$, $v = 0.99995c$, $v = 0.9999995c$? Use the approximations you've just developed, as appropriate. If you find yourself rounding off and saying that $y = 1$ or $y = 0$, you've rounded too much!

e. (15 points) A muon is a particle very much like an electron, only with more mass. Unlike electrons, muons are unstable; they have a half-life of only 2.2×10^{-6} seconds, after which they decay into an electron, a muon neutrino, and an anti-electron neutrino. Very fast-moving muons are produced in the upper atmosphere of the Earth (i.e., 100 km above the Earth's surface) in collisions with high-energy cosmic rays (produced ultimately from distant supernova explosions), and come whizzing down to be detected on the surface. A typical atmospheric muon is moving at 0.9999995 the speed of light; i.e., $(1 - 5 \times 10^{-7})c$. Taking into account time dilation, how far could such a muon travel before decaying (i.e., before one half-life is over)? Now do the same calculation, not taking into account time dilation. Discuss: does the fact that we observe muons that were produced 100 km away give any support to Einstein's prediction of time dilation?

2. Antimatter! (10 points)

Two 10-metric-ton trucks (each with mass = 10^4 kilograms), each going 100 kilometers per hour in opposite directions have a head-on collision. Each truck has a kinetic energy of $\frac{1}{2}mv^2$ according to Isaac Newton. Since the velocity is small relative to the speed of light, Newton's formula is quite accurate. When the trucks collide, the energy of the resulting explosion is equal to the total original kinetic energy of the two trucks. Now suppose that a 10-metric-ton truck made out of matter collides with a 10-metric-ton truck made out of anti-matter, so that the two trucks annihilate, causing an explosion in which the total mass of the two trucks is converted into energy according to Einstein's formula $E = mc^2$. Assume that the matter and anti-matter trucks hit at such a low velocity that their kinetic energy can be ignored. Calculate the ratio of the energy of collision of the matter and anti-matter trucks to that of the two normal trucks in the head-on collision described above. If you see a matter and an anti-matter truck about to hit each other, should you try to get far away?

3. What's at the center of our Galaxy? (15 points)
You observe a star on a circular orbit around the center of our Galaxy. The radius of the orbit is 20 light-days. From the Doppler shift of its absorption lines you determine that its orbital speed is 1,000 km/s. Based on your measurement, what would be your estimate for the mass of the object that the star is orbiting? What kind of an object do you think this is? Give arguments to support your conclusion. A movie of orbits of stars observed near Galactic center can be seen at http://tinyurl.com/5ck5rp for your entertainment.

4. The Hubble Constant (30 points)
Astronomers at Princeton are leading the Sloan Digital Sky Survey (SDSS) – a survey that mapped positions and distances of a million galaxies using a dedicated 2.5 meter telescope in New Mexico. You can get more information and images from SDSS at `http://www.sdss.org/iotw/archive.html` . In this problem, you will use data from this survey to calculate the Hubble Constant. The attached figure shows the spectrum of a star in our galaxy and spectra of four distant galaxies, as measured by this project. For each of the galaxies we indicate the measured brightness, in units of Joules per square meter per second. Assume that each of them has the same luminosity as that of the Milky Way (10^{11} times the luminosity of the Sun, or 4×10^{37} J/s). This is a problem in which the use of a calculator is appropriate.

a. (10 points) Determine the distance to each of the four galaxies, using the inverse-square law relation between brightness and luminosity. Express your answers both in meters and in Megaparsecs (1 Mpc $= 3.1 \times 10^{22}$ meters), and give two significant figures.

b. (10 points) The spectrum of each of these objects shows a pair of strong absorption lines of Calcium, which have rest wavelength $\lambda_0 = 3935$ Ångstrom and 3970 Ångstrom, respectively. The wavelengths of these lines in the galaxies have been shifted to longer wavelengths (i.e., redshifted), by the expansion of the universe. As a guide, the spectrum of a star like the Sun is shown in the upper panel; the calcium lines are at zero redshift.

Measure the redshift of each galaxy. That is, calculate the fractional change in wavelength of the Calcium lines. *Hint: The tricky thing here is to make sure you're identifying the right lines as Calcium. In each case, they are a close pair; for Galaxy #2, they are the prominent absorption dips between 4100 and 4200 Ångstroms. Measure the redshift for both of the Calcium lines in each galaxy (and in each case, the two lines should give the same redshift, of course!). In this problem, please give your final redshift to two significant figures. Do the intermediate steps of the calculation without rounding; rounding too early can result in errors.*

c. (10 points) Given the redshifts, calculate the velocity of recession for each galaxy, and in each case use the distances to estimate the Hubble Constant, in units of kilometers per second per Megaparsec. You will not get identical results from each of the galaxies, due to measurement uncertainties (but they should all be in the same ballpark), so average the results of the four galaxies to get your final answer.

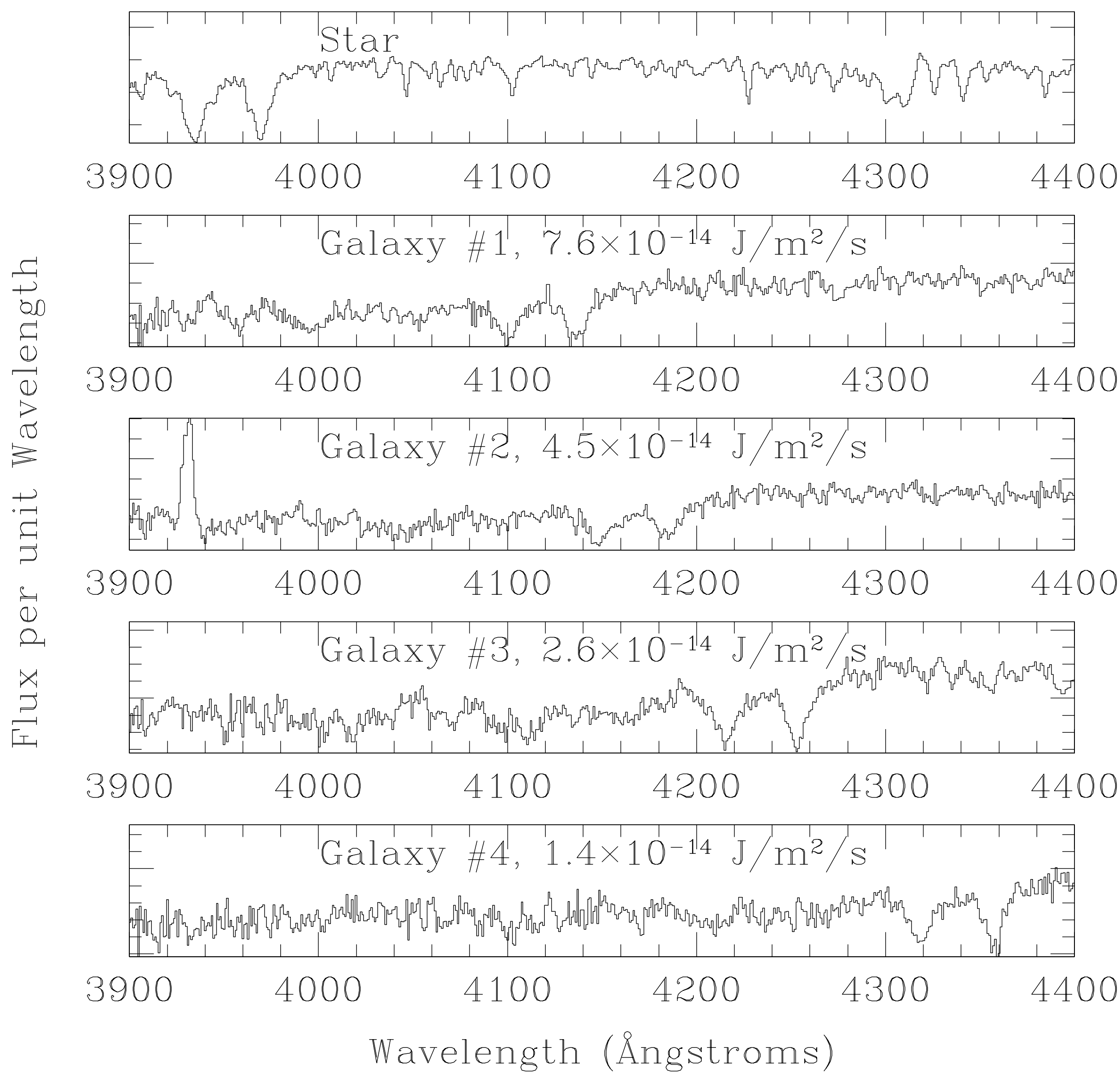

Figure for Problem 4.