Solutions to Homework #4, AST 203, Spring 2012

Due in class (i.e., by 4:20 pm), Thursday April 5

*General grading rules: One point off per question (e.g., 3a or 3b) for egregiously ignoring the admonition to set the context of your solution. Thus take the point off if relevant symbols aren't defined, if important steps of explanation are missing, etc. If the answer is written down without *any* context whatsoever, take off 1/3 of the points. One point off per question for inappropriately high precision (which usually means more than 2 significant figures in this homework). No more than two points off per problem for overly high precision. Three points off for each arithmetic or algebra error (although if the part of the problem in which this arithmetic error is made is worth only two points, then take only two points off). Further calculations correctly done based on this erroneous value should be given full credit. However, if the resulting answer is completely ludicrous (e.g., 10^{-30} seconds for the time to travel to the nearest star, 50 stars in the visible universe), and no mention is made that the value seems wrong, take three points off. Answers differing slightly from the solutions given here because of slightly different rounding (e.g., off in the second decimal point for results that should be given to two significant figures) get full credit. Two points off per question for not being explicit about the units, and one point off for not expressing the final result in the units requested. Leaving out the units in intermediate steps should be pointed out, but no points taken off. Specific instructions for each problem take precedence over the above. In each question, one cannot get less than zero points, or more than the total number the question is worth.*

100 total points

1. Transits of extrasolar planets (30 points)

NASA mission called "Kepler" was launched in 2009 to observe the transits of extrasolar planets. During a transit, a planet goes in front of its host star as seen from our vantage point on Earth. This blocks a little bit of light from the star, so an accurate measurement of the change in brightness of the star will show a temporary dimming. The main goal of the Kepler mission is to detect Earth-size planets in the habitable zone around stars like our Sun. In this problem, we will calculate the magnitude of the effect that the Kepler mission observes. Consider a planet with the radius of the Earth on a circular orbit with radius of 1 AU about a star that has parameters like our Sun. The orbital plane of this planet lies along our line of sight.

a) How long does a transit last (in hours)? (*10 points*)

Hint: the Earth is very far from this planetary system.

Solution:

The duration of the transit is determined by how long it takes the planet to cross the face of the star. Since we are observing this from a vantage point very far away, the duration is just the diameter of the star divided by the orbital speed of the planet (if we were observing this transit closer to the star we would have to worry about the finite angular size of the star, and when exactly the planet starts crossing the limb of the star). We have found the orbital speed in a Keplerian orbit several times before. You can derive it from using Kepler's third law, and the fact that orbital velocity is the circumference of the orbit divided by the period:

$$v = \sqrt{\frac{GM_\odot}{r}}.$$

Substituting numbers for the solar mass and 1AU for the distance, we get the orbital speed of 30 km/s. The duration of the transit is then $2R_\odot/v = 2 \times 7 \times 10^5$ km/(30 km/s) $\approx 5 \times 10^4$ s = 13 hours.
Full credit for simply quoting the orbital speed, or just calculating it as $2\pi \times 1$ AU/1year.

b) By how many percent does the brightness of the host star diminish in the middle of the transit? (*10 points*)
Solution:
In the middle of the transit the planet blocks the area on the surface of the star equal to the *cross sectional area* of the planet, $A_{\rm planet} = \pi R_{Earth}^2$ (note, that this is not the total surface area of the planet!). The unperturbed brightness of the star is proportional to the area of the face of the star $B_{*0} \propto A_* = \pi R_*^2$ (note that this is also cross-sectional area). In the middle of the transit, the area of the face of the star is reduced to $B_{*1} \propto A_* - A_{\rm planet}$. The fractional change in brightness B_* of the star as seen by us will then be

$$\frac{\Delta B_*}{B_*} = \frac{B_{*0} - B_{*1}}{B_{*0}} = \frac{A_* - (A_* - A_{\rm planet})}{A_*} = \frac{A_{\rm planet}}{A_*} = \left(\frac{R_{planet}}{R_*}\right)^2.$$

Substituting $R_{\rm Earth}$ and $R_\odot$, we get the change of $(10^{-2})^2 = 10^{-4}$, or 0.01%.
3 points off for using the total surface areas ($4\pi R^2$). 5 points for recognizing how to do the problem but completely failing in the algebra. 2 points off for answers of 99.99% – not the change in brightness, but the final brightness.

c) Consider now the planet the size of Jupiter, orbiting at 0.5 AU from the same star. By how many percent does the brightness of the host star diminish in the middle of the transit by this planet? How long does this transit last? (*10 points*)
Hint: the Earth is still very far from this planetary system. **Solution**
We are observing this transit from very far away, so the distance between the planet and the host star will not affect the answer. The difference is just the cross sectional area of the planet. Jupiter's radius is 11 times the Earth radius, so the answer in b) will increase by $11^2 \sim 100$. So, the answer is 1% change in brightness. This change can be detected even with the Earth-based telescopes, and this method has been used to find many "hot Jupiters" – gas planets orbiting close to other stars. *(6 points for this part)*

For the duration of this transit, we can scale from the solution of 1a). The speed of a planet at 0.5 AU from the star is going to be different by $1/\sqrt{1/2} = \sqrt{2}$ compared to orbit at 1AU. The diameter of the star is still the same, so the crossing time is going to be $\sqrt{2}$ times shorter, or about 9 hours. *(4 points for this part)*

2. White Dwarfs and Neutron Stars (30 points)
White dwarfs and neutrons stars are "compact objects" – stars whose large mass (near a

solar mass) is concentrated in a small volume. Such stars possess exotic properties that allow us to probe the behavior of matter at extreme densities. Consider a white dwarf (WD) with the mass of 1 $M_\odot$ and radius 1 $R_{\rm Earth}$, and a neutron star (NS) with a mass of $1.4 M_\odot$ and radius 10 km.

a) Find how much mass is contained in one teaspoon (1cm^3) of material from WD and NS. Express the answer in metric tons. (*7 points*)
Hint: the material can be taken to be at the mean density of these stars
Solution
To calculate the mass of the teaspoon worth of NS material, we start with computing the mean density of the star. We know that 1.4 $M_\odot$ is confined within a sphere of radius 10 km:

$$\rho_{\rm NS} = \frac{M_{\rm NS}}{\text{Volume of NS}} = \frac{1.4 M_\odot}{4\pi r^3/3} = \frac{1.4 \times 2 \times 10^{30} \text{ kg}}{4 \times 3 \times 10^4 \text{ m}^3/3} = 7 \times 10^{17} \text{ kg/m}^3.$$

The mass in the teaspoon is then $\rho_{\rm NS} \times V_{\rm teaspoon} = 7 \times 10^{17} \text{kg/m}^3 \times 10^{-6} \text{m}^3 = 7 \times 10^{11}$ kg $\approx$ 1 billion tons. For the WD, we use the same formula, but plug in $1 M_\odot$ for the mass and 6400 km for the radius. We find:

$$\rho_{\rm WD} = \frac{M_{\rm WD}}{\text{Volume of WD}} = \frac{1. M_\odot}{4\pi r^3/3} = \frac{1.4 \times 2 \times 10^{30} \text{ kg}}{4 \times 3 \times 6.4 \times 10^6 \text{ m}^3/3} = 2 \times 10^9 \text{ kg/m}^3$$

The mass in the teaspoon is then $\rho_{\rm WD} \times V_{\rm teaspoon} = 2 \times 10^9 \text{kg/m}^3 \times 10^{-6} \text{m}^3 = 2 \times 10^3$ kg $\approx$ 2 tons.

b) Calculate the mass density of a neutron and compare it to the mean density of NS. A neutron can be considered as a sphere of radius 1 femto-meter (10^{-15} m). (*6 points*)

Solution:
Using the same argument to find the mean density of the neutron as for the star above we get:

$$\rho_{\rm neutron} = \frac{m_n}{4\pi r_n^3/3} = \frac{1.7 \times 10^{-27} \text{ kg}}{4 \times 3 \times (10^{-15} \text{ m})^3/3} = 4 \times 10^{17} \text{ kg/m}^3$$

Note that the mean density of the NS is very close to the nuclear density! You should not be worried that the numbers don't match *exactly* – this is an order of magnitude estimate! Being close to nuclear density means that the neutrons are almost touching each other inside the NS. This is the most dense configuration of matter we know of – anything denser than this would collapse into a black hole.

c) Calculate the mean distance between atoms in WD. Express your result in Ångstroms (1 Ångstrom = 10^{-10} meters). You may approximate the white dwarf to be made entirely of carbon atoms (12 times the mass of hydrogen). Compare with the typical sizes of atoms under normal conditions, ~ 1 Ångstrom. *Hint: We are looking for an approximate answer here.* (*7 points*)

Solution
To calculate the mean distance between atoms in WD, we need to ask for the volume

which one carbon atom finds itself in inside the star. To do this, we need to know the number of Carbon atoms in the WD and the volume of the WD. The number can be obtained from the mass of the star ($M_{\rm WD}$) and the mass of one Carbon atom ($M_{\rm C} = 12m_p$, where m_p is the mass of the proton).

$$N_C = M_{WD}/M_C = 2\times10^{30}\ \text{kg}/(12\times1.6\times10^{-27}\ \text{kg}) \approx 1/10\times10^{57} = 10^{56}\text{atoms}.$$

Note, that not every calculator will be able to deal with numbers this big, so adding powers of 10 by hand may be necessary! The volume per atom is then $(4\pi R^3_{\rm WD}/3)/N_C = 10^{21}\ \text{m}^3/(10^{56}) = 10^{-35}\ \text{m}^3$. Let us approximate each atom as occupying a cube of side L and thus volume $V = L^3$, so the mean distance between atoms is roughly L (ignoring a factor of 2):

$$L = (10^{-35})^{1/3}\,\text{m} \approx 2\times10^{-12}\,\text{m} = 0.02\,\text{Å}.$$

This is 50 times smaller than the size of the atom under normal conditions; the atoms are tremendously squeezed in a white dwarf! In fact, the matter inside the white dwarf is not in atomic state – it is a sea of carbon nuclei and electrons, a plasma. However, when electrons are packed this close together, they start to exert the quantum mechanical "degeneracy" pressure, and this is what supports the white dwarf against collapse.
The comparison with the size of ordinary atoms is worth 2 points. Any reasonable calculation gets full credit. No points off for missing factors of 2.

d) We can estimate the rate of rotation of compact objects by knowing that they are the end product of contraction of rotating main sequence stars after they exhaust their nuclear fuel. Each piece of gas in a star contracts in such a way that the product of its distance to the rotation axis times the velocity of rotation about the axis is a constant. (*Hint: we really mean the velocity of rotation, not angular velocity here*). This is known as conservation of "angular momentum," and is the same phenomenon that causes a spinning ice skater to turn faster when she raises her hands. Consider a point on the equator of a the core of the star before the collapse. The radius of the core is half the radius of the Sun and its rotation period is 30 days (like it is for the Sun). Now imagine that this point contracts with the core and ends up on the surface of a neutron star. Find the expected rotation period of such a neutron star (in milliseconds). (*10 points*)

Solution
Conservation of angular momentum implies that the product of velocity of rotation of the point on the equator, v, times the equatorial radius R is constant, or $v\times R = const$. We can express the rotation velocity of a point on the equator in terms of the period of rotation, P, or $v = 2\pi R/P$. Substituting back, we get $R^2/P = const$. Therefore, if the radius shrinks, the period would have to become shorter as well (the core spins up). Let's denote the original period and radius with the subscript CO for core, and final period and radius with subscript NS (for neutron star). Then we have

$$\frac{R^2_{\rm CO}}{P_{\rm CO}} = const = \frac{R^2_{\rm NS}}{P_{\rm NS}}.$$

Expressing the period of the neutron star from this equation, we get:

$$P_{\rm NS} = \left(\frac{R_{\rm NS}}{R_{\rm CO}}\right)^2 P_{\rm CO}.$$

Checking the sanity of the formula, we see that since $r_{\rm NS}/r_{\rm CO} \ll 1$, the period of rotation of the neutron star will be much shorter, as we expect. Substituting numbers, $R_{\rm NS}/R_{\rm CO} = 10\text{km}/(0.5R_\odot) \approx 2\times10^{-5}$. The period is then $P_{\rm NS} = (2\times10^{-5})^2\times 30\text{days} \approx$ 1 millisecond. This number is actually near the limit of how fast a neutron star can rotate. From observations of radio pulsars we infer that neutron stars are born with periods from 1 to 20 milliseconds and then spin down with time.
3 points for calculating the period assuming $\omega r = const$, which is incorrect.

3. Neutrinos Coursing Through Us (25 points total).
In 1987, the astronomical world was electrified with the news of a supernova exploding in the Large Magellanic Cloud, a dwarf galaxy companion to the Milky Way, at a distance of 150,000 light years. It was the nearest supernova to have gone off in 400 years, and was studied in great detail. Its luminosity was enormous; the explosion released as much visible light energy in a few weeks as the Sun will emit in its entire lifetime of 10^{10} years. It was easily visible to the naked eye from the Southern hemisphere. However, theories of the mechanisms taking place in the supernovae predict that the visible light represents only 1% of the total energy of the supernova; there is 100 times more energy emitted in the form of neutrinos, in a blast lasting only a few seconds! When the supernova was discovered, the late John Bahcall of Princeton and his colleagues did the calculation that follows, asking the question whether any of the neutrinos emitted from the supernova should have been detected here on Earth.

a. (7 points) From the information given, calculate the total amount of energy emitted by the supernova in neutrinos. Express your answer in Joules.
Answer: We are told that the energy emitted by the supernova in visible light is equal to that emitted by the Sun in 10^{10} years. We can look up the luminosity of the Sun (energy emitted per second), and simply multiply by the 10^{10} years:

$$\text{Total Energy emitted} = 4\times10^{26}\,\text{Joules/second}\times10^{10}\,\text{years}\times\frac{3\times10^7\,\text{seconds}}{1\,\text{year}} \approx 10^{44}\,\text{Joules}.$$

The energy associated with the neutrinos is 100 times larger still than that, namely 10^{46} Joules.
Three points off for forgetting the factor of 100 for the neutrinos.

b. (5 points) Each neutrino has an energy of roughly 1.5×10^{-12} Joules. Calculate how many neutrinos are emitted by the supernova. (This is an easy calculation, but will give you a very large number!).
Answer: If each neutrino has an energy of 1.5×10^{-12} Joules, the total number of neutrinos emitted by the star is straightforward:

$$10^{46}\,\text{Joules}\times\frac{1\,\text{neutrino}}{3/2\times10^{-12}\,\text{Joules}} = 2/3\times10^{58}\,\text{neutrinos} = 7\times10^{57}\text{neutrinos}$$

c. (8 points) Scientists have built a number of detectors of neutrinos (in order to look for neutrinos from the Sun, as we discussed in class). One of the largest is called Kamiokande in Japan. In 1987, it consisted of a cube of water roughly 10 meters on a side (it has since been expanded). Calculate how many neutrinos from the supernova would have passed through the cube of water. *Hint: the inverse square law holds for neutrinos just as it does for light. Consider the area of the detector, and remember that the neutrinos are sent out in all directions from the supernova; at the time that the neutrinos hit Kamiokande, they are spread out over an enormous sphere centered on the supernova.*

Answer: These neutrinos are emitted essentially all at once, and thereafter, travelling at the speed of light, they expand into a huge spherical shell of ever-increasing radius. Thus by the time they impinge on the Earth, they are spread out over a spherical shell of radius 150,000 light years. The number density on the shell is:

$$\frac{\text{Number}}{\text{Surface Area}} = \frac{7 \times 10^{57}\,\text{neutrinos}}{4\,\pi(1.5 \times 10^{5}\,\text{ly} \times 10^{16}\,\text{m/ly})^2} = \frac{1/2 \times 10^{57}\,\text{neutrinos}}{2 \times 10^{42}\text{m}^2} = 2.5{\times}10^{14}\,\text{neutrinos/m}^2$$

That is, every square meter on the Earth's surface was peppered with 250 trillion neutrinos from the supernova!

The neutrino detector is 10 meters on a side, and therefore has a surface area of 100 square meters. Therefore 2.5×10^{16} neutrinos passed through it.

Give no more than 3 points for the incorrect calculation involving not the areas of the detectors and the sphere, but rather the volume. Treat as an arithmetic error (3 points off) calculations using the full surface area of the cube (600 m^2).

d. (5 points) Neutrinos are ghostly particles; they are very difficult (but not completely impossible) to detect. A typical neutrino can pass through the entire Earth without anything happening to them. However, it turns out that roughly 1 in 10^{15} (i.e., one in a quadrillion!) of the neutrinos passing through Kamiokande will interact with the water there, and thereby be detected. Calculate how many neutrinos should have been detected by Kamiokande.

Answer: We know how many neutrinos passed through Kamiokande, and we know that one in 10^{15} will be detected. Therefore, the total number of neutrinos detected is simply:

$$2.5 \times 10^{16}\text{neutrinos} \times \frac{1\,\text{neutrino detected}}{10^{15}\,\text{neutrinos}} = 25\,\text{neutrinos}.$$

This differs by only a factor of two of the number of neutrinos actually seen from the 1987 supernova.

When the discovery of the supernova was first announced, John Bahcall and his colleagues Arnon Dar and Tsvi Piran immediately realized the possibility that Kamiokande could have detected the neutrinos from it. They locked themselves in their office, took the phone off the hook, did essentially the calculation that you've just done, and sent a paper off to the journal Nature, all within 24 hours. They wanted to make a prediction about the neutrinos, untainted by any news that the neutrinos actually were found. Indeed, a few days later, the news about the Kamiokande detection came out. The Bahcall et al. paper was published on 1987 March 12 (the supernova itself went off on February 23), and has the following understated but triumphal final sentence: "Note added in proof: Since this paper was received on 2 March, the neutrino burst was found by the Kamiokande experimental group, with properties generally consistent with the calculated expectations."

We end with a famous poem, written in 1960. The poem states that neutrinos have no mass. It has recently been discovered (using the same Kamiokande experiment described above!) that they in fact do have a very small mass; each neutrino has a mass of roughly 6×10^{-8} that of an electron. This discovery (together with the detection of neutrinos from the supernova) was recognized in the 2002 Physics Nobel Prize. John Bahcall, mentioned above, was one of the people responsible for this discovery, but was passed over for the prize...

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Neutrinos, they are very small.
They have no charge and have no mass
And they do not interact at all.
The earth is just a silly ball.
To them, through which they pass,
Like dustmaids down a drafty hall
Or photons through a sheet of glass.
They snub the most exquisite gas,
Ignore the most substantial wall,
Cold-shoulder steel and sounding brass,
Insult the stallion in his stall,
And, scorning barriers of class,
Infiltrate you and me! Like tall
And painless guillotines, they fall
Down through our heads into the grass.
At night, they enter at Nepal
And pierce the lover and his lass
From underneath the bed - you call
It wonderful: I call it crass.

 -- John Updike, ``Cosmic Gall''
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4. True or False? (15 points)
Determine if the following statements are true or false, and give the reasoning to support your conclusion in a short paragraph.

a) If the Sun had been born as a high-mass star some 4.5 billion years ago, rather than as a low-mass star, the planet Jupiter would probably have Earth-like conditions today, while the Earth would be hot like Venus. (*5 points*)
Solution

This statement is false. If the Sun had been born as a high-mass star 4.5 billion years ago, it would have exploded as a supernova a long time ago.
2 sympathy points for missing the age problem, and carrying on an argument about a hotter star producing warmer temperatures on Earth and Jupiter.

b) An open cluster that contains many O stars is unlikely to also have lots of white dwarfs. (*5 points*)
Solution
This statement is true. Since the open cluster still has O stars, it is fairly young, and its lower mass stars that could give rise to white dwarfs have not yet had enough time to evolve. It takes about 10 billion years for a sun-like star to go through its life and make a white dwarf. O stars live less than 10 million years.

c) If a $3M_\odot$ main-sequence star is in a binary with a 2.5 $M_\odot$ red giant, the red giant must have been more massive than $3M_\odot$ when it was a main-sequence star. (*5 points*)
Solution
This statement makes sense. The $2.5M_\odot$ red giant had to be more massive than its companion at some point in the past in order for it to be more advanced in its evolutionary state than its companion. This is because the main sequence lifetime directly depends on mass. Two stars that are in a binary likely formed at the same time, so if the more massive one now is less evolved than the less massive one, this could only be if the $2.5M_\odot$ star used to be more massive and has lost some mass after its main sequence evolution (most likely it transferred the mass to the companion, or lost it to a wind). *(2 points only if the fact that the two stars are in a binary and must be the same age is not used at all).*

In these problems, a correct answer without explanation or with completely egregious explanation gets 1 point.