

Homework #3, AST 203, Spring 2010

Due in class (i.e. by 4:20 pm), **Tuesday March 8**

*General grading rules: One point off per question (e.g., 1a or 1b) for egregiously ignoring the admonition to set the context of your solution. Thus take the point off if relevant symbols aren't defined, if important steps of explanation are missing, etc. If the answer is written down without *any* context whatsoever (e.g., for 1a, writing "164.85 years", and nothing else), take off 1/3 of the points. One point off per question for inappropriately high precision (which usually means more than 2 significant figures in this homework). However, no points off for calculating a result to full precision, and rounding appropriately at the last step. No more than two points per problem for overly high precision. Three points off for each arithmetic or algebra error. Further calculations correctly done based on this erroneous value should be given full credit. However, if the resulting answer is completely ludicrous (e.g., 10^{-30} seconds for the time to travel to the nearest star, 50 stars in the visible universe), and no mention is made that the value seems wrong, take three points further off. Answers differing slightly from the solutions given here because of slightly different rounding (e.g., off in the second decimal point for results that should be given to two significant figures) get full credit. One point off per question for not being explicit about the units, or for not expressing the final result in the units requested. Two points off for the right numerical answer with the wrong units. Leaving out the units in intermediate steps should be pointed out, but no points taken off. Specific instructions for each problem take precedence over the above. In each question, one cannot get less than zero points, or more than the total number the question is worth.*

100 total points

1. Greenhouse on Titan (30 points)

This question asks you to calculate temperatures and the implications of those temperatures for Saturn's large moon Titan. Titan (radius 2,576 km) is slightly larger than the planet Mercury and is the only moon in the solar system with a dense atmosphere – its atmospheric surface pressure is 50% greater than Earth's, in fact. Nevertheless, it is far from the Sun...

- a) (10 points) Saturn orbits 9.6 AU from the Sun, and Titan orbits Saturn. Treat Titan as having, on average, the same distance from the Sun as Saturn. Neglecting the greenhouse effect for now, write down an equation that balances the power (or, energy per time) from the Sun absorbed by Titan with the power radiated away from Titan, treating Titan as a black body with albedo A . Use symbols for luminosity L of the Sun, distance from Sun to Titan, d , radius of Titan, r , and its temperature T . Do not yet fill in any values for your variables.

Solution: The brightness of the Sun as seen at Titan is given by the inverse square law:

$$b = \frac{L}{4\pi d^2}.$$

This is the amount of energy per unit time *per unit area* received at Titan. The effective cross-sectional area of Titan is πr^2 , and thus the total energy per unit time

falling on Titan is the product of the brightness and that area, namely:

$$L \frac{r^2}{4d^2}.$$

But only a fraction $1 - A$ of that light is absorbed, the rest is radiated. So the incoming power is:

$$(1 - A)L \frac{r^2}{4d^2}.$$

Now we need to balance this incoming power with the radiated power. Titan is emitting like a black body, so its luminosity is $\sigma T^4 4\pi r^2$, where we take into account that it is emitting through the whole surface area. The final balance equation is then:

$$(1 - A)L \frac{r^2}{4d^2} = \sigma T^4 4\pi r^2$$

- b) (10 points) Take Titan's albedo to be 0.22. Assuming no greenhouse atmosphere, what is the expected black-body temperature (in Kelvin) of Titan from the equation you set up in part a)? Take the solar luminosity to be 3.9×10^{26} W.

Solution. We can solve the balance equation above for the temperature of Titan:

$$T = \left((1 - A)L \frac{r^2}{4d^2} / \sigma 4\pi r^2 \right)^{\frac{1}{4}} = \left((1 - A) \frac{L}{16\pi d^2 \sigma} \right)^{\frac{1}{4}},$$

or plugging in numbers:

$$T = \left((1 - 0.22) \frac{4 \times 10^{26} \text{ W}}{16\pi (9.6 \times 1.5 \times 10^{11} \text{ m})^2 \times (5.7 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4})} \right) = 85 \text{ K}$$

- c) (2 points) Titan's atmosphere is mostly nitrogen, like Earth's, but contains nearly 2% methane – a powerful greenhouse gas. Assume Titan's atmosphere provides an ideal greenhouse. We saw in lecture that this would be expected to increase the surface temperature of a world by a factor of $2^{1/4}$ or about 1.2. If Titan's atmosphere were a perfect greenhouse, what would you expect Titan's surface temperature to be?

Solution: For a perfect greenhouse, the temperature would be $2^{1/4}$ times larger, or 101 K (100K is ok too).

- d) (2 points) Earth's average temperature is 288 K. Extreme temperatures at the poles and equator vary about this average by as much as 90 K lower and higher, respectively. Assume that Titan sees a temperature variation (about its surface greenhouse temperature) from its poles to equator proportionally the same as Earth's. What are the minimum and maximum temperature on Titan likely then to be?

Solution: The key word in this problem is "proportionally". For earth 90K is $\approx 30\%$ of 288K, so we expect variation by 30% around the mean. This would put the minimum and maximum temperature on Titan to be $\sim 70\text{K}$ and $\sim 130\text{K}$.

- e) (6 points) Download and print the file “methane phase diagram” (Fig. 1) from the Blackboard site under Course Materials. Titan’s surface pressure is 1.5 bar, but on the logarithmic pressure scale of Fig. 1, this can be well approximated as 1.0 bar, written as “1.0E0”. Using a pencil and ruler, and referring to the temperature scale on the x-axis, draw a horizontal line on this plot indicating the temperature range that could be experienced on Titan’s surface. What phases of methane are possible across this temperature range? On Earth, a meteorological cycle is driven as water moves between solid, liquid, and vapor phases. Is a meteorological cycle, but with methane instead of water, possible on Titan?

Solution: At 1.0 bar pressure, the graph of methane phase diagram indicates that methane can go from solid phase at 70K to liquid phase at 90 to 110 K, and then become vapor at higher temperatures. So, the minimum and maximum temperatures that we find (70 and 130K) give us a range of all possible methane phases. This means that a meteorological cycle with methane is possible on Titan. (Of course, on Earth evaporation depends on humidity, wind, and temperature, and occurs at temperatures well below the phase transition at 373 K from water to vapor. But this problem suggests that just as Earth has temperatures that allow three phases of water somewhere on its surface, Titan should have temperatures allowing three phases of methane.)

Problems 2-5. A New Denizen of the Solar System

The discovery of the dwarf planet “Eris” in 2005 threw the astronomical community into a tizzy and made international headlines; it is slightly larger than Pluto (as we’ll see) and brought up interesting questions about what the definition of a planet is. Eventually, this resulted in the controversial demotion of Pluto from the 9th planet of the Solar System to just one of a number of dwarf planets.

As a top AST 203 student, you have been named to head a team of astrophysicists to calculate some of properties of Eris. For the next four problems of this homework, you are going to be using the same tools that professional astronomers used to understand the properties of this object. Throughout, assume that Eris is spherical and is observed at opposition (i.e., the Earth lies on the straight line connecting the Sun and Eris).

2. (15 points) First, you will determine how far Eris is. In five hours, Eris is observed to move 7.5 arcseconds relative to the background stars as seen from Earth; see <http://www.astro.princeton.edu/ast203/eris.gif>. Because (as we’ll see) Eris is much further from the Sun than is the Earth, it is moving quite a bit slower around the Sun than the Earth, so this apparent motion on the sky is essentially entirely *parallax* due to the Earth’s motion. First, calculate the speed with which the Earth goes around the Sun, in kilometers/second. Then use this information and the small-angle formula to calculate the distance from the Earth to Eris. Express your result in AU. Compare with the semi-major axis of Pluto’s orbit (which you’ll need to look up). *Hint: it will help to draw a diagram.*

Solution: As the Earth moves, the direction to which we point to Eris changes. In order to do calculations; we need to know how fast the Earth travels around the Sun. This can be done either by plugging into the formula for orbital speed we derived in earlier homework, or more simply by remembering that the Earth travels the circumference of the Earth’s

orbit in the time of one year, thus the speed is given by the distance divided by the time:

$$v \approx \frac{2\pi \times 1.5 \times 10^8 \text{ km}}{3 \times 10^7 \text{ sec}} = 30 \text{ km/s}$$

to one significant figure (I made the standard approximation that $\pi \approx 3$).

At this speed, how far does Earth move in 5 hours (18,000 seconds)? Rounding to 20,000 seconds, the answer is $l = 600,000$ kilometers. The parallax diagram is a long skinny triangle, with the 600,000 km of the Earth's path at the base, the distance d to Eris its length, and a $7.5''$ angle at its apex. We could use fancy notions from trigonometry to solve this triangle (remember, it is d that we're after here), but it is far easier to use the small-angle formula. If we measure angles θ in *radians*, and if the angles are small (as they often are in astronomy), there is a very simple relationship between the long side d and short side l of long skinny triangles:

$$\theta = \frac{l}{d} \tag{1}$$

Indeed, this is how radians are defined. For those of you who know some trig, you will remember that radians are defined as an arclength of a circle, divided by its radius. If the arc is small enough, it is essentially straight, and we're essentially looking at a long skinny triangle.

OK, we're almost done. In our case, we know θ , $7.5''$, and we know l , about 6×10^5 km. We want to solve for d ; we get: $d = l/\theta$. But we are given θ in arcseconds; we need it in radians for this formula to work. There are 2π radians in a full circle (remember, it is arclength divided by radius; for a full circle, this means the circumference divided by the radius, or 2π). A full circle is 360 degrees, so the conversion from degrees to radians is $2\pi/360 \approx 1/57$. Similarly, there are 3600 arcseconds in a degree. Thus the conversion from arcseconds to radians is $1/57/3600 \approx 5 \times 10^{-6}$. This is a useful number to remember; there are roughly 200,000 arcseconds in a radian. So:

$$7.5'' = 7.5'' \times 5 \times 10^{-6} \text{ radians}'' \approx 4 \times 10^{-5} \text{ radians.} \tag{2}$$

So the distance d to Eris is therefore:

$$d = \frac{l}{\theta} = \frac{6 \times 10^5 \text{ km}}{4 \times 10^{-5}} \approx 1.5 \times 10^{10} \text{ km.} \tag{3}$$

There are 1.5×10^8 km in an AU, so the distance to Eris is 100 AU. (For this problem, we can take the distance from the Sun to Eris, and from the Earth to Eris, as essentially the same).

The semi-major axis of Pluto's orbit is 40 AU; Eris is *quite* a bit more distant. Interestingly, Eris is on a highly elliptical orbit, $e = 0.44$ (much larger than any of the other planets in the Solar System) and is currently very close to aphelion. Its perhelion is actually within the orbit of Pluto, but it won't be there for another 280 years, given its 560-year orbital period.

Four points for the calculation of the speed of Earth's orbit, doing either from Newton's laws, or as done here. Three points for properly understanding the mechanics of parallax, but not going any further. Two points for blindly plugging into distance = 1 pc/parallax,

without any further thinking. Three points off for factors of two errors due to thinking that the 7.5 arcsec number refers to half the angular shift. Two points off for not comparing to Pluto. Here I've given answers to one significant figure, but full credit for doing things consistently to two or three decimal places.

3. (30 points) We're next going to calculate how big Eris is, from its apparent brightness. We'll have to do a few calculations along the way. Eris shines in two ways: from its reflected light from the Sun (which will be mostly visible light), and from its blackbody radiation from absorbed sunlight (which will mostly come out as infrared light); it will be important in the calculations that follow to keep in mind which of the two you're dealing with. The albedo of Eris (i.e., the fraction of the sunlight incident on Eris that is reflected) is very high, about 85%. This suggests that Eris is covered by a layer of shiny ice; spectroscopy tells us that the ice is composed of frozen methane, CH_4 .

(a) (10 points) We are going to start by deriving an expression for the brightness of Eris which depends on its distance from the Sun d , and its radius r . First, calculate the amount of sunlight reflected by Eris per unit time (i.e., its luminosity in reflected light); express your answer in terms of d , r , the albedo A , and the luminosity of the Sun L (you will find this similar to problem 1, but with one important difference). Don't plug in numbers yet!

Solution: The brightness of the Sun as seen at Eris is given by the inverse square law:

$$b = \frac{L}{4\pi d^2}.$$

This is the amount of energy per unit time *per unit area* received at Eris. The cross-sectional area of Eris is πr^2 , and thus the total energy per unit time falling on Eris is the product of the brightness and that area, namely:

$$L \frac{r^2}{4d^2}.$$

But only a fraction A of that light is reflected, the rest is absorbed. So our final answer is:

$$AL \frac{r^2}{4d^2}.$$

Note that this is very similar to problem 1, but we used different albedo prefactor, because here we are interested in the reflected and not absorbed light.

4 points off for neglecting the factor A . 3 points off for using the term $1 - A$ instead of A . Two points off for mistakes at the level of writing $4\pi r^2$ for the cross-sectional area of Eris. Five points off for ignoring the instructions, and plugging numbers in before getting to the final algebraic expression.

(b) (10 points) We detect only a tiny fraction of this light reflected by Eris. Calculate the brightness, via the inverse square law, of Eris as perceived here on Earth. Again, don't plug in numbers yet; just do algebra; again your answer will involve d , r , A , and L . (*Hint: With Eris so distant from the Sun, the tiny 1 AU separation between the Earth and the Sun is negligible. So make the approximation that the distances from the Sun to Eris, and from Eris to the Earth, are the same.*)

Solution: What we have calculated in part (a) is the luminosity (in reflected light, at least), of Eris. We are observing it a distance d away (again, we're taking the approximation

that the distance from the Sun to Eris, and from Eris to the Earth, are essentially the same). So its brightness just follows from the inverse square law, namely:

$$\text{brightness } b = AL \left(\frac{r}{2d} \right)^2 \times \frac{1}{4\pi d^2} = AL \frac{r^2}{16\pi d^4}. \quad (4)$$

The brightness of a distant asteroid falls off as the inverse *fourth* power of the distance. This is why these distant guys took so long to be discovered; they are really faint!

Assign partial credit as in part (a); full credit for answers consistent with part (a). Strictly speaking, the calculation above assumes that the reradiated light is spread over a full sphere, while in fact it is reradiated over only a hemisphere. Full credit, therefore, for answers that express the brightness as twice the value given above.

(c) (10 points) OK, now for some numbers. The measured brightness of Eris is 2.4×10^{-16} Joules meters⁻² second⁻¹. We know the distance d to Eris and its albedo A , and can look up the luminosity of the Sun L . Use this information to solve for the radius r of Eris. (*Hint: be careful to be consistent with your units!*) Compare with the radius of Pluto (1200 km). This is what led to the controversy of what a planet is: if Pluto is considered a planet, then certainly Eris should be as well. We'll write much more about this controversy, and some of the related issues, in the solution set to this homework.

Solution: To do this, let's solve our equation for r :

$$r = 4d^2 \sqrt{\frac{\pi b}{AL}} \quad (5)$$

Let's plug in numbers. We have to be careful to consistently use MKS units here. $d = 100AU = 1.5 \times 10^{13}$ meters.

$$r = 4 \times (1.5 \times 10^{13})^2 \times \sqrt{\frac{\pi \times 2.4 \times 10^{-16}}{0.85 \times 4 \times 10^{26}}} \text{ meters} \quad (6)$$

Let's see how we can do this without a calculator. I'm going to approximate $0.85 \times 4 = \pi$, so the quantity under the square root is roughly 2.4×10^{-42} , and its square root is about 1.5×10^{-21} . So

$$r = 4 \times 2.25 \times 10^{26} \times 1.5 \times 10^{-21} \text{ meters} \quad (7)$$

$4 \times 2.25 = 9$, and 9×1.5 is roughly 15. So we get a radius of 15×10^5 meters, or 10^6 meters, or 1500 kilometers. This is seriously large, larger than Pluto itself!

The discovery of Eris (named after the goddess of strife and discord in Greek mythology) set off a controversy in the astronomical community about whether it should be called a planet, and what the definition of a planet is. Reams have been written on this subject. The basic problem is that the concept of a planet has evolved as we have learned more, and we now realize that things that might conceivably be called planets now fall into a variety of categories:

- "Terrestrial planets", relatively small rocky objects in the inner solar system, including Mercury, Venus, Earth and Mars;

- “Gas giants”, much much larger and more massive bodies in the outer part of the solar system, including Jupiter, Saturn, Uranus and Neptune (many have argued that Uranus and Neptune should be in their own category of “ice giants”; they are actually mostly frozen gas, not vaporous gas like Jupiter and Saturn”
- “Dwarf Planets”, some found in the main asteroid belt between Mars and Jupiter, but a class to which Pluto, Eris, and other recent discoveries belong.

And this list doesn’t yet include the massive moons of the Earth, Jupiter, and Saturn (the largest of which are considerably larger than Pluto), or the planets discovered around other stars. The term “planet” is now too broad to allow a single, all-encompassing clean definition, and our field has become richer with the discovery of Eris and its brethren. Wikipedia has some good articles on the subject, if you’d like to learn more.

2 points off for not including a comparison with Pluto’s radius.

4. (10 points) Calculate the angular size of Eris (i.e., the angle the diameter of Eris makes on the sky). Compare this to the resolution of the Hubble Space Telescope (i.e., the smallest things it can see as extended), which is about 40 milli-arcseconds; will you be able to resolve Eris (i.e., will it look like a point of light or a finite-size object in a telescope)?

Solution The diameter of Eris is $D_E \approx 3000\text{km}$, and the distance is $d = 100\text{AU}$. Imagining the triangle covering the diameter of Eris on one end, and Earth at the other vertex, the angular size of Eris is then

$$\theta = D_E/d = 3000 \text{ km}/100 \text{ AU} = 2 \times 10^{-7} \text{ radians} \approx 40 \text{ milliarcseconds.}$$

We used here conversion $1 \text{radian} \approx 200,000 \text{arcseconds}$. This is at the very limit of the resolution of Hubble, so you can resolve it (in fact, HST did).

Angular size calculation is worth 8 points, comparison to HST 2 points. Take one point off for not converting angular size to arcseconds. No points off if the calculation comes out to be a bit lower than HST resolution and they conclude that it cannot be resolved. It’s a tricky measurement anyways.

5. (15 points) You go ahead and observe Eris with Hubble, and find that it has a moon orbiting it! Observations with Hubble show that this moon (called Dysnomia) makes an almost circular orbit around Eris with a period of 15.8 Earth days. The semi-major axis of the orbit subtends an angle of $0.53''$ as seen from Earth. Calculate the semi-major axis in kilometers, and calculate the mass of Eris in kilograms. Compare with the mass of Pluto ($1.3 \times 10^{22} \text{ kg}$). Is Eris more massive?

Solution: This is a simple application of Newton’s form of Kepler’s Third Law, which relates the period and semi-major axis of an orbiting body to the mass of the object it orbits:

$$P^2 = \frac{4\pi^2 a^3}{GM}$$

Here we are given the period and the semi-major axis, and we need to calculate the mass. Solving for the mass gives:

$$M = \frac{4\pi^2 a^3}{GP^2}$$

But do we have the semi-major axis? What we're given is the angle the semi-major axis subtends in our Hubble Space Telescope images. Another opportunity to use the small-angle formula. Consider a very long skinny triangle, with length given by the distance from the Earth to Eris (100 AU) and interior angle $0.53''$; we want to find the base of the triangle. It is:

$$s = \theta d = 0.53 \text{ arcsec} \times \frac{1 \text{ radian}}{2 \times 10^5 \text{ arcsec}} \times 1.5 \times 10^{10} \text{ kilometers} = 37,000 \text{ kilometers.}$$

If we plug this in, then, to Kepler's Third Law for the mass, we find:

$$M = \frac{4\pi^2 \times (3.7 \times 10^7 \text{ meters})^3}{2/3 \times 10^{-10} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1} \times (15.8 \text{ days} \times 86,400 \text{ sec/day})^2}.$$

At this point, I will admit that I reached for my calculator, and got a value of 1.6×10^{22} kg. This is remarkable: Eris is a bit more massive than is Pluto!

Calculation of the semi-major axis is worth 3 points. Writing Kepler's Third Law in a form that yields a mass, and going no further, is worth 3 points. Comparison with Pluto's mass is worth 1 point.