Solutions to Problem Set #1, AST 203

Due on February 16, 2012

General grading rules: One point off per question (e.g., 2a or 2b) for egregiously ignoring the admonition to write in full sentences. One point off per question for inappropriately high precision (which usually means more than 2 significant figures in this homework). No more than 2 total points per problem off for not writing in full sentences, and two points per problem for overly high precision. Three points off for each arithmetic or algebra error. Further calculations correctly done based on this erroneous value should be given full credit. However, if the resulting answer is completely ludicrous (e.g., 10^{-30} seconds for the time to travel to the nearest star, 50 stars in the visible universe), and no mention is made that the value seems wrong, take three points off. One point off per question for not being explicit about the units, or for not expressing the final result in the units requested. Specific instructions for each problem take precedence over the above. In each question, one cannot get less than zero points, or more than the total number the question is worth.

1. Scientific notation review *10 total points (2-2-2-2-2)*

Write the following in proper scientific notation. Full sentences are not required here.

a) Four hundred seventy million
Answer: 4.7×10^8
The answer 4.70×10^8 gets full credit. Equivalent answers, such as 470×10^6, get full credit. More zeroes added on get one point off. Rounding to 5×10^8, or 4×10^8, gets no credit. The answer of 470,000,000 (i.e., not writing in scientific notation) gets no credit.

b) Seven trillion, two hundred seventy-one billion
Answer: 7.271×10^{12}.
Rounded-off answers get no credit.

c) One two hundredth
Answer: 5×10^{-3}.

d) The speed of light is 3.0×10^5 km/sec. What is the speed of light in units of cm/sec?
Answer: $3.0 \times 10^5\,\text{km/s} \times 10^5\,\text{cm/km} = 3.0 \times 10^{10}\,\text{cm/sec}$.
No credit if the conversion between cm and km is not made explicit. One point off for writing the answer as 3×10^{10} cm/sec, i.e., with only one significant figure.

e) The number $\pi \times 13$ (don't forget the second bullet above about correct precision)
Answer: We are given two significant figures in the number 13, so our result should also be to two significant figures. The correct answer is 41 (rounded up from 40.8), or in scientific notation, 4.1×10^1. In this case it is worth keeping the extra digit in π during the calculation, and rounding at the end.
Either form is fine. Answer of 41 is also accepted. No credit for the wrong number of significant figures, as that is the entire point of the problem.

2. Looking out in space and back in time *20 total points (3-3-5-5-4)*

Take the speed of light to be 3.0×10^5 km/sec. Take one astronomical unit (AU) to be 150 million kilometers.

a) To correct precision, how far back in time are we looking when we look at the Sun from the Earth (give your answer in minutes); (*3 points*)
Answer: The time is the distance (1 AU) divided by the speed (the speed of light). In this case, this gives:

$$\frac{1.5 \times 10^8 \, \mathrm{km}}{3.0 \times 10^5 \, \mathrm{km/s}} = 500 \, \mathrm{sec},$$

a number accurate to two significant figures (as both the input numbers in this calculation have two significant figures). Note that I didn't have to use a calculator for this, as $1.5/3 = 0.5$, certainly a calculation I can do in my head, and the rest of the calculation is just keeping track of powers of ten.

But we are asked to give the answer in minutes. There are 60 seconds in a minute, so the calculation is straightforward:

$$500 \, \mathrm{sec} \times \frac{1 \, \mathrm{minute}}{60 \, \mathrm{sec}} = 8.3 \, \mathrm{min},$$

correct to two significant figures. Note that the number of 60 seconds in a minute is *exact*, and therefore this does not change the number of significant figures in your answer.
One point off for giving the answer as 8 minutes, to a single significant figure.

b) How far back in time are we seeing the star Betelgeuse, which is 643 light years away? Give your answer in years. (*3 points*)
Answer: This is easy, if we remember that the speed of light year is one light year per year, *exactly*. So light takes 4.3 years to travel 4.3 light years.
Full credit for doing this (correctly) the hard way, involving converting light years to kilometers, and seconds into years. If they lose a significant figure in the process (e.g., by using 1 ly = 1×10^{13} km), and explain that the final answer should therefore be rounded to one significant figure, then give full credit.

c) Suppose you were trying to have a conversation with an astronaut on the Moon. The Moon is 384,400 km from Earth. If you sent a radio signal to an astronaut on the Moon, and she or he replied immediately, how long would be the gap between when you sent the signal and you received your answer?
Solution: *9 points* Here we are interested in the round-trip time, which is the time light takes to travel $384,400 \times 2$ kilometers. This time is:

$$384,400 \, \mathrm{km} \times 2 \times \frac{1}{3.0 \times 10^5 \, \mathrm{km/sec}} = 2.6 \, \mathrm{seconds}.$$

Here we give the answer to two significant figures, which is the precision with which we're given the speed of light. When ground control in Houston communicated with astronauts on the Moon, this delay of 2.6 seconds was quite noticeable.

Three points off for forgetting the factor of 2. No points off for significant figures if 4 significant figures are given, a more exact value is given for the speed of light, and an explicit discussion of the significant figures is given.

Now imagine that you were an engineer for the Mars Exploration Rover spacecraft scooting around on the surface of Mars. Suppose you need to send it an emergency transmission to prevent it from driving into a ditch. Mars is on a roughly circular orbit of radius 1.5 AU around the Sun. The Earth is also on a circular orbit, of radius 1.0 AU, of course. (For these two planets at the level of precision asked for in this problem, the approximation of their orbits as circles is a good one.) About how long would it take your message to reach the rover, when:

d) Mars is in opposition, i.e., when the Earth lies on the line between Mars and the Sun? (*5 points*)
Solution: The orbits of Mars and the Earth are in the same plane, and they are closest when Mars is in *opposition*, i.e., when the Earth lies on the line between Mars and the Sun, and is between them. It is then 1.5 - 1.0 = 0.5 AU between Mars and the Earth. The time required for light to travel that distance is then:

$$0.5\,\mathrm{AU} \times 1.5 \times 10^8 \frac{\mathrm{km}}{\mathrm{AU}} \times \frac{1}{3.0 \times 10^5\,\mathrm{km/sec}} \times \frac{1\,\mathrm{min}}{60\,\mathrm{sec}} = 4\,\mathrm{minutes},$$

where we give our answer to one significant figure. *Two significant figures is acceptable as well.*

e) Mars is farthest from Earth in the two planets' orbits? (*4 points*)
Solution: When Mars is in *conjunction*, then the Sun lies between it and the Earth, and Mars is $1 + 1.5 = 2.5$ AU away, 5 times further. This is a light travel time of 21 minutes (*again, one or two significant figures are fine*).

This substantial delay is a real issue in communications between Earth and Mars. *Spirit* and *Opportunity* are two robotic rovers that explored the surface of Mars, but they could not be operated in real time by remote control because of this time delay. Instead, they were designed to be quite autonomous: once a day they are uploaded with detailed instructions of what to do for the day, and the rest of the time they are on their own, with software clever enough to figure out what to do if they hit an unanticipated rock.

3. The density of water, air, and the Galaxy *40 points total (10-10-10-10)*

A cubic meter of air at sea level contains approximately 3×10^{25} molecules. To a fair approximation, all the molecules are N_2, i.e. a pair of nitrogen atoms bound together into a diatomic molecule. (Note that the part of the air that you need to breathe, i.e. the oxygen (O_2) molecules, comprises only about 21% of the air by volume; we ignore this O_2 fraction here and pretend that the whole atmosphere is N_2.) Each N atom has a mass 14 times that of hydrogen (H).

a) Calculate the mass density (kilograms per cubic meter) of air on Earth. Do this calculation without a calculator, show your work, and use scientific notation. In this and in all parts of this problem, give the final answer using the appropriate amount of precision. You will need the mass of a hydrogen atom (to a good approximation the same as the mass of a proton, 1.6×10^{-27} kg). *(10 points)*

Answer: The density is equal to the mass of a cubic meter of air. That mass is equal to the number of molecules of N_2, times the mass per molecule. A single molecule consists of 2 atoms of N, and thus has a mass 28 times that of hydrogen (whose mass is sometimes referred to as an Atomic Mass Unit, or AMU). Thus:

$$\text{Density of air} = 3 \times 10^{25}\,\frac{\text{molecules}}{\text{meter}^3} \times \frac{28\text{AMU}}{\text{molecule}} \times 1.6 \times 10^{-27}\,\frac{\text{kg}}{\text{AMU}}$$

Before you reach for that calculator, remember that we need an answer to a single significant figure. Here's one way to do the arithmetic without a calculator: Remember than $1.6 \approx 5/3$. The 3's cancel, leaving:

$$5 \times 28 \times 10^{-2}\,\text{kg/meter}^3 \approx 1.5\,\text{kg/m}^3.$$

If we round to a single significant figure, we get either 1 or 2 kg/m^3.

Full credit for either answer, or any answer with two significant figures (but not more!) in between 1 and 2 kg/m^3. Using a mass of 14 AMU rather than 28 AMU for Nitrogen should be treated like an arithmetic error; take off three points.

b) To a fair approximation, the total volume of Earth's atmosphere can be treated as lying within 8 km of the surface of the planet. (Of course the atmosphere extends much higher, but it becomes exponentially less dense with increasing height; in fact about 60% of the atmosphere lies within 8 km of the surface.) The Earth is a sphere of radius 6,400 km. Using the results of a), calculate the mass of the Earth's atmosphere in kilograms. You may use a calculator if you wish, but give answers to the proper amount of precision in scientific notation, and show your work. *Hint: the volume of a thin spherical shell is the surface area of the sphere times the thickness of the shell.* (*10 points*)

Answer: We're told that the volume in question is the surface area of the Earth, times the thickness, thus:

$$\text{Volume} = 8\,\text{km} \times 4 \times \pi \times (6400\,\text{km})^2$$

Let's try this without a calculator, again knowing that the answer need be accurate to only one significant figure (as we are given the shell thickness to a single figure). $64^2 \approx 4000$, to a single figure, so we get:

$$\text{Volume} \approx 8 \times 4 \times 3 \times 4 \times 10^3 \times 10^4\,\text{km}^3 \approx 4 \times 10^9\,\text{km}^3.$$

The mass of an object (here, the Earth's atmosphere) is the product of its volume and its density. This is easy, but it is vitally important that the units match, so you need to convert kilometers to meters.

$$\text{Mass} = \text{Density} \times \text{Volume} = 1.5\,\text{kg/m}^3 \times 4 \times 10^9\,\text{km}^3 \times 10^9\text{m}^3/\text{km}^3 = 6 \times 10^{18}\,\text{kg}.$$

With 1000 kilograms in a ton, that is 6 quadrillion (10^{15}) tons, an impressive number! *Full credit for numbers consistent with those above.*

c) How many tons of air are over your head when you walk outside? (*Hint: you may find it helpful to estimate or measure the circumference of your head.*) Do you feel the weight of this air? (*10 points*)

Here we want to calculate the mass of air in a column with the height of the atmosphere (8km as said above) and the cross-sectional area of your head, which we need to estimate. One of the easiest ways to measure the cross-sectional area of the head is to remember that the head is roughly a circle in section, hence by measuring its circumference we can compute its radius, r, and then the area as πr^2. A typical person has roughly 22 inches of head circumference, or 22in $\times$ 2.54cm/in $\approx$ 55cm (perhaps you may find it worthwhile to make this measurement in the beginning and the end of AST203 to check if your head expands over the semester...). The radius of the head is then $r = 55\text{cm}/(2\pi) \approx 10$ cm. The cross-sectional area of the head is $\pi \text{r}^2 \approx 300\text{cm}^2$. The mass contained in the column is:

$$\text{Mass} = \text{Density} \times \text{Volume} = 1.5\,\text{kg/m}^3 \times 300\text{cm}^2 \times 8\text{km}$$

Note that we have meters, centimeters and kilometers in one expression. Let's convert them all to meters, using $1\text{cm} = 10^{-2}\text{m}$ and $1\text{km} = 10^3\text{m}$:

$$\text{Mass} = \text{Density} \times \text{Volume} = 1.5\,\text{kg/m}^3 \times (300\text{cm}^2 \times 10^{-4}\text{m}^2/\text{cm}^2) \times 8 \times 10^3\text{m}$$

$$= 360\text{kg} \approx 0.4\,\text{tons}$$

That's a lot of air over your head! Do you feel it? Yes! The weight of this air is the cause of the atmospheric pressure that we experience at sea level.
Numbers in the right ballpark get full credit. Any reasonable answer on the question of atmospheric pressure is accepted (it's worth 4 points). Saying that we don't feel this air is also ok, if it's justified that we feel differences in air pressure, not the pressure itself. Our bodies are at atmospheric pressure on the inside (more or less), so it's not that apparent. However, when you try to open a jar with a metal lid that was vacuum sealed, the reason it's so hard is because the atmospheric pressure is pushing the lid in.

e) The Milky Way Galaxy contains about 10^{11} stars with mass similar to that of the Sun, and has a radius of roughly 50,000 light years. Approximating the Milky Way Galaxy as a sphere, compute its mean density in kilograms per cubic meter, and compare with (i.e. take the ratio with) the density you calculated for Earth's air. Do this calculation without a calculator, show your work, and use scientific notation. Do you find the result surprising? *(10 points)*

Answer: To calculate the density, we need to calculate both the mass and the volume of the Milky Way. We are told that the Milky Way has 10^{11} stars, each with the mass of the Sun (2×10^{30} kg), so the total mass is the product of the two, or 2×10^{41} kg. For the volume, we are told the Milky Way is a sphere (at least approximately!), with radius 50,000 light years. We are going to need an answer using meters in the end, so we should convert to meters right away. We said in class that there are 10^{13} km/ly, and we know that there are 10^3 m/km, so there are 10^{16} m/ly. So the radius of the Milky Way is 5×10^{20} meters. Now, the volume of a sphere of radius r is $(4\pi/3)r^3$ (which we will approximate as $4\,r^3$), giving us a volume:

$$\text{Volume} = 4 \times (5 \times 10^{20}\,\text{meters})^3 = 5 \times 10^{62}\,\text{m}^3.$$

Wow, the numbers are getting *seriously* big! The density is then the ratio of the mass to the volume, giving:

$$\text{Density} = \frac{2 \times 10^{41}\,\text{kg}}{5 \times 10^{62}\,\text{m}^3} = 4 \times 10^{-22}\,\text{kg/m}^3.$$

This is an astonishingly low number. Despite containing one hundred billion stars, our Milky Way is mostly empty space, and the mean density is enormously smaller than that of ordinary air. We take the ratio with the density of air to find:

$$\frac{4 \times 10^{-22}\,\text{kg/m}^3}{1.5\,\text{kg/m}^3} = 3 \times 10^{-22}.$$

Calculating either the mass of the Milky Way, or the volume of the Milky Way alone is worth 3 points. Calculating the ratio with the density of air is worth 2 points. Full credit for calculating the ratio of the air density to the galaxy density (and getting an answer of 3×10^{21}*).*

4. Kepler's third law *20 total points*

Kepler's third law, or the "harmonic law," says that the square of the orbital period P of a body orbiting the Sun is proportional to the cube of the body's semimajor axis a. If P is measured in years and a is measured in AU, the proportionality constant is $1\,\text{year}^2/\text{AU}^3$, and in these units Kepler's third law is: $P^2 = a^3$. Consider the following data for three objects orbiting our Sun. In a-c), how long (in years) does it take each of these bodies to complete one orbit? You may use a calculator for these problems.

a) Venus has a semimajor axis of 0.72333 AU; (*4 points*)

b) The asteroid Ceres has a semimajor axis of 2.766 AU; (*4 points*)

c) Jupiter's orbit around the Sun has a semi-major axis 5.20 times larger than that of the Earth. (*4 points*)

Answer: In each case, we are given the semi-major axis and asked to solve for the period. Kepler's third law gives $P = a^{1.5}$, with a in AU and P in years. We're given quite a few significant figures in each case, so a calculator is the way to proceed. Plugging in numbers, and making the significant figures match in each case, gives:

a) Venus: Period of 0.61518 years.

b) Ceres: Period of 4.600 years.

c) Jupiter: Period of 11.9 years.
No points off for getting the number of significant figures off by one, in either direction. Full credit for doing it (correctly) the hard way, (i.e., using MKS units). Small differences in the answer in the last significant figure are fine.

d) Using the result of c), what is Jupiter's orbital velocity, in km/s? (*8 points*)
Answer: To a very good approximation, Jupiter is on a circular orbit. Then we know the radius of the orbit of $a = 5.20$ AU and the time it takes to go around the circle is $P = 11.9$ years from c). The speed then is $v = 2\pi a/P$, or the circumference of the circle over the period.

$$\mathrm{v} = \frac{2\pi \times 5.20\,\mathrm{AU}}{11.9\,\mathrm{years}} = \frac{2\pi \times 5.20\,\mathrm{AU} \times 1.50 \times 10^{8}\,\mathrm{km/AU}}{11.9\,\mathrm{years} \times 3.15 \times 10^{7}\,\mathrm{sec/years}} = 13.2\,\mathrm{km/s}$$

4 points for figuring out what to do, 2 for the calculation. Small differences in the answer in the last significant figure are fine.

5. Order of Magntitude Estimate. (10 total points)
Most of the homework problems in AST203 will be self-contained, meaning that you will be given all the information needed to solve the problem in the problem text. In the real world, however, we often encounter problems, which are not well defined, where not all the information is given, and some assumptions have to be made to get an answer. To make a first pass at such problems it is very useful to make an educated rough estimate of the unknowns, often to the nearest power of 10, and proceed with a quick calculation to get a "ballpark" estimate of the answer. Such "order of magnitude" (OOM) estimates are very common in science, and can be extremely useful in everyday life as well. A good example of an OOM estimate is the classic Fermi problem (named after Enrico Fermi, who was fond of such calculations): "How many piano tuners are there in Chicago?" You can read its solution on Wikipedia: `http://en.wikipedia.org/wiki/Fermi_problem` .

The problem for you is this: *Estimate how many roses are bought on Valentine's day in the US.* Please show your assumptions and estimates. The precise number you get is less important than your reasoning behind it, explaining how you estimated it.

Solution.
There are roughly 300 million people in the US. The average life expectancy in the US is about 75 years. Let's say people are capable of buying roses from 18 years of age until about 70. This leaves us with $\sim 70\%$ of the life span in the rose-buying capacity. If the number of people of every age bracket is the same (i.e., there are as many 5-10 year olds as there are 40-45 year olds – this is not necessarily true, but this is an estimate), then there

are $0.7 \times 300 = 210$ million people of the rose-buying age. Half of them are men, and they are the majority of buyers on Valentine's day, so that's 105 million. Obviously, not every man in this age group is in a relationship, or remembers the date. Let's say half are in a relationship at any given time and will not forget to do something about a gift. That's a bit over 50 million. Let's say half of that will buy not roses, but some other type of flower or card, or chocolate, etc. This leaves us with 25 million rose-shopping men. Considering that a single rose stem can go for upwards of $4 on Valentine's day, it is safe to assume that not every one will be able to afford a dozen roses. Let's say people buy 7 roses on average (some buy a single rose, some a dozen, some more). We are then looking at 25 million times $7 \sim 180$ million roses.

This is just the number bought by men. It is safe to assume that women will also buy some roses on Valentine's day. That can easily add another 25 million up to about 200 million, or 2×10^8 roses. This estimate probably underestimates the contribution of women and overestimates the percentage bought by men.

Anything with reasonable justification and not just one guess at the final number gets full credit. Numbers above 3 billion get 3 points off. This would assume that everyone in the US bought a dozen roses on Valentine's day – clearly this is unreasonable.